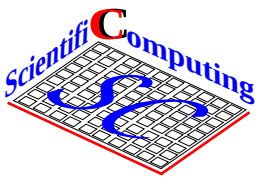

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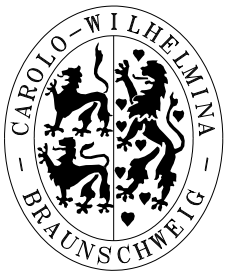

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Abstract:
The present review discusses recent developments in numerical techniques for the solution of systems with stochastic uncertainties. Such systems are modelled by stochastic partial differential equations (SPDEs), and techniques for their discretisation by stochastic finite elements (SFEM) are reviewed. Also, short overviews of related fields are given, e.g. of mathematical properties of random fields and SPDEs and of techniques for high-dimensional integration.

After a summary of aspects of stochastic analysis, models and representations of random variables are presented. Then mathematical theories for SPDEs with stochastic operator are reviewed.

Discretisation-techniques for random fields and for SPDEs are summarised and solvers for the resulting discretisations are reviewed, where the main focus lies on series expansions in the stochastic dimensions with an emphasis on Galerkin-schemes.

Finally, numerical methods required in the solution of SPDE and the post-processing of results are discussed.

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Contents

Chapter 1

Introduction

1.1 Overview

In the recent years, there has been an increased interest in the simulation of systems with uncertainties. This work reviews and classifies recent developments in this field.

The interest in uncertain systems stems from the fact that uncertainties remain in most models of real world problems. Uncertainties arise either due to our lack of knowledge (epstisemic uncertainties), or due to intrinsic variabilities of physical quantities (aleatoric uncertainties), e.g. due to heterogeneities in materials. Data like domain geometry, material properties, or loads, are usually not known perfectly. Due to the uncertainties in the model, it is uncertain to what degree the prognoses of numerical simulations match reality; this fact is often ignored in traditional engineering practice.

Clearly, it is desirable to quantify the uncertainties in the answer, and different approaches have been proposed for this. Let us briefly mention a selection:

- Uncertainties may be described by stochastic models—uncertain parameters are then described by random variables, uncertain time dependent functions are represented by stochastic processes, and uncertain spatial properties are modelled by random fields. If the physical system is described by a partial differential equation (PDE), then the combination with the stochastic model results in a stochastic PDE (SPDE). The numerical solution of SPDEs is the focus of the present review.

- Alternatively, fuzzy sets may be used to describe uncertainties; for an introduction see e.g. Kruse et al. (1995). They describe parameters by possibility functions specifying their degree of belonging to a set. Maglaras et al. (1997) compare random and fuzzy models of uncertainty and state

that uncertainty is better represented by a stochastic description if enough statistical information is available and that otherwise fuzzy theory is better suited.

- In contrast to fuzzy and stochastic methods, set methods are independent of a probability or possibility measure. They assume that parameters are inside given sets. Then they compute sets in that the response is guaranteed to lie. Representatives of this approach are interval analysis (e.g. Alefeld and Mannheimer, 1974) and its generalisations to ellipsoidal and convex modelling (e.g. Elishakoff, 1999a).

- A related issue are intrinsic heterogeneities of materials. To obtain effective material parameters, homogenisation methods may be used (e.g. Torquato, 2000; Zohdi and Wriggers, 2001), but if a separation of scales is not possible, then stochastic averaging may be necessary. Recent developments in this direction are discussed in section 3.4.3.

This review concentrates on stochastic models. These have been used in several fields, e.g. in structural engineering (e.g. Haldar and Mahadevan, 2000; Kuireghian and Ke, 1988; Bucher et al., 2000; Ghiocel and Ghanem, 2002) or in the earth sciences (e.g. Ripley, 1988; Christakos, 1992; Dagan and Neuman, 1997). They have a sound mathematical foundation, and the advance in computational methods and the increase in computational power has made their application to complex systems feasible.

Stochastic models require information on the statistics of system properties. This is sometimes seen as a disadvantage, as exact statistics are hard to obtain (e.g. Elishakoff, 1999b). But if the available information is scant, uncertainties may be modelled by ad-hoc assumptions, while at the same time all available statistical information can fully be used.

The question whether stochastic models are valid for modelling uncertainties may be answered either by philosophical reasoning or by comparing their prognoses with reality; see Natke and Ben-Haim (1997); Christakos (1992); Elishakoff (1999c) and the references therein for in-depth discussions on the validity of stochastic models. See Maglaras et al. (1997) for a validation of stochastic models by experiments.

Stochastic mechanics is a fast growing area of research, and some reviews were published:

- Matthies et al. (1997) reviewed stochastic finite elements (SFEM) with an emphasis on structural stochastic problems.

- Schuëller (1997) edited a state of the art report on computational stochastic mechanics with contributions from many authors.

- Sudret and Kiureghian (2000) published a review with an emphasis on reliability assessment, including tutorials and comparisons of SFEM methods.

The present report extends and updates the aforementioned reviews in several directions: it gives an overview of the mathematical foundations of stochastic fields and their discretisation, summarises the mathematical theory of stochastic partial differential equations (SPDEs), and reviews methods for their discretisation. For numerical methods the emphasis is on stochastic Galerkin methods, but other techniques are discussed also. A brief overview of numerical procedures and of post-processing concludes the review.

1.2 Terms and Symbols

A consistent notation is used throughout the text for all reviewed publications.

1.2.1 Glossary

The following abbreviations are used:

CC Clenshaw-Curtis (quadrature rule).

CDF Cumulative Distribution Function (of a random variable).

Deterministic Code
See deterministic solver.

Deterministic Solver
It is sometimes assumed that the spatial discretisation is performed by some existing simulation software, e.g. by a finite element code. This software is called the *deterministic solver* or the *deterministic code* throughout the text.

DOF Degree of Freedom.

FE Finite Element.

FEM Finite Element Method.

FORM First Order Reliability Method.

KL Karhunen–Loève (e.g. KL-expansion, KL-series).

PDE Partial Differential Equation.

PDF Probability Density Function.

RF Random field.

RV Random variable.

SFEM Stochastic Finite Element Method.

SORM Second Order Reliability Method.

SPDE Stochastic Partial Differential Equation.

1.2.2 Notation and Conventions

The following conventions are used:

$\boldsymbol{u}$ Vectors are small letters in a bold italic font.

$\mathbf{u}$ Block vectors are small letters in a bold upright font.

$\boldsymbol{K}$ Matrices are capital letters in a bold italic font.

$\mathbf{K}$	Block matrices are capital letters in a bold upright font.
γ, κ, ξ	Random variables and random fields are in Greek letters.
$\boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\xi}$	Random vectors are in bold Greek letters.
$\alpha, \beta, \gamma, \iota$	These Greek letters are used for multi-indices.
$f^{(\alpha)}$	A superscript multi-index in round brackets denotes the coefficient of a random variable in its polynomial chaos expansion.
$f^m_{\leq k}$	The projection of the random variable f on the m-dimensional polynomial chaos of degree k.

1.2.3 Symbols

The following symbols are used throughout the text:

$\|\cdot\|_p$	Standard L^p-norm (section 2.2.1).
$\|\cdot\|_\infty$	Standard L^∞-norm, essential supremum (section 2.2.1).
$\|\cdot\|_{\rho,r}$	Hida distribution and test function norms (section 2.3).
$\|\cdot\|_\rho$	Kondratiev distribution and test function norms (section 2.3).
$\langle\cdot,\cdot\rangle$	Duality pairing (section 2.2.2).
$(\cdot,\cdot)$	Scalar product (section 2.2.2).
$\|\alpha\|$	Modulus of a multi-index α, defined as $\|\alpha\| = \sum_{i\in\mathbb{N}} \alpha_i$ (section 2.2.3).
$\alpha!$	Factorial of a multi-index α, defined as $\alpha! := \prod_{i\in\mathbb{N}}(\alpha_i!)$ (section 2.2.3).
$\odot$	The Wick-product (section 4.2.2).
$\alpha, \beta, \gamma, \iota$	Multi-indices, exception: γ is sometimes a Gaussian random variable (section 2.2.3).
$\mathcal{B}$	Probability space, σ-algebra of events. (section 2.1)
χ_B	Characteristic function of a set B.
$C_c^\infty(R)$	Space of infinitely often differentiable functions with compact support in $R \subset \mathbb{R}^d$.
$\operatorname{cov}(\kappa_1, \kappa_2)$	Covariance of the random variables κ_1, κ_2, defined as $\operatorname{cov}(\kappa_1, \kappa_2) = \mathbf{E}\left((\kappa_1 - \mu_{\kappa_1})(\kappa_2 - \mu_{\kappa_2})\right)$ (section 2.1)
$\boldsymbol{C}_{\boldsymbol{\kappa}}$	The covariance matrix of a random vector $\boldsymbol{\kappa}(\omega)$.
d	Dimension of space; the spatial domain of the SPDE is $R \subset \mathbb{R}^d$ (section 3.1).
Δ	The Laplace-operator.
$\operatorname{erf}(x)$	The distribution function of a standard Gaussian random variable, $\operatorname{erf}(x) := F_{\mathcal{N}(0,1)}(x)$ (section 2.1).
$\mathbf{E}(\cdot)$	Expectation operator, $\mathbf{E}(g(\kappa)) = \int_\Omega g(\kappa(\omega))\, dP(\omega) = \int_{\mathbb{R}} g(\kappa)\, dF_\kappa(\kappa)$ (section 2.1).

$f_\gamma(x)$	The probability distribution function of a Gaussian random variable, $f_\gamma(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ (section 2.1).
$F_\kappa(k)$	Distribution function of a real-valued random variable κ, defined as $F_\kappa(k) = P_\kappa(-\infty, k) = P\{\kappa < k\}$ (section 2.1).
γ	Depending on the context, γ may be a multi-index (section 2.2.3). Alternatively, γ may denote a Gaussian random variable or a Gaussian random field $\gamma(x,\omega), x \in R, \omega \in \Omega$. (section 3.2.1).
Γ_m	Gaussian probability measure in m dimensions, $d\Gamma_m(x) = (2\pi)^{-m/2} \exp(-\lvert x\rvert^2/2)\, dx$ (section 2.2.1).
$h_i(x)$	Univariate Hermite-polynomial of degree i for $x \in \mathbb{R}$ (section 2.2.3).
$H_\alpha(\omega)$	Multivariate Hermite-polynomial, indexed by the multi-index α. Defined as $H_\alpha(\omega) = \prod_{i\in\mathbb{N}} h_{\alpha_i}(\omega_i)$. (section 2.2.3).
$H_0^1(R)$	Sobolev Hilbert space of once differentiable functions, completion of $C_c^\infty(R)$ (section 4.1).
$\mathcal{H}$	A Hilbert space, the spatial part of the solution (section 4.1).
$\mathcal{H}_{:=p:}$	Homogeneous chaos of degree p (section 2.2.3).
$\mathcal{H}^m_{:=p:}$	m-dimensional homogeneous chaos of degree p (section 2.2.3).
$\mathcal{H}_{:\le p:}$	Polynomial chaos of degree p (section 2.2.3).
$\mathcal{H}^m_{:\le p:}$	m-dimensional polynomial chaos of degree p (section 2.2.3).
$I_m(\psi)$	Integral in an m-dimensional space $I_m(\psi) = \mathbf{E}(\psi(\omega))$ (section 7.1).
$k_1, k_2, \ldots$	The eigenfunctions $k_1(x), k_2(x), \ldots$ in the Karhunen–Loève-expansion, $k_i \in L^2(R)$ (section 5.1.2).
κ	A random field $\kappa(x,\omega), x \in R, \omega \in \Omega$ (section 3.1).
$\boldsymbol{\kappa}$	Vector of random variables. If $\kappa(x,\omega)$ is a random field, then $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_m)$ is the vector of random variables in its truncated KL-expansion (section 5.1.2).
κ_i	The uncorrelated random variables $\kappa_i(\omega)$ occurring in the Karhunen–Loève-expansion of a random field $\kappa(x,\omega)$ (section 5.1.2).
$\kappa^{(\alpha)}$	Projection of a random field onto the polynomial chaos, $\kappa^{(\alpha)}(x) = \mathbf{E}(\kappa(x,\cdot)H_\alpha)$
$\lambda_1, \lambda_2, \ldots$	The eigenvalues in the Karhunen–Loève-expansion, $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$ (section 5.1.2).
$L^p(V)$	For $0 < p < \infty$ the standard-Banach spaces (or Hilbert-space for $p = 2$) of functions $\in V$ whose p-th exponent is finitely integrable. $L^\infty(V)$ is the space of essentially bounded functions (section 2.2.1).
μ_κ	The mean, $\mu_\kappa = \mathbf{E}(\kappa)$, of the random variable κ (section 2.1).
$\boldsymbol{\mu}_{\boldsymbol{\kappa}}$	The mean vector of the random vector $\boldsymbol{\kappa}$.

m	Denotes the number of independent random variables.
$\mathcal{N}(\mu,\sigma^2)$	A Gaussian random variable with mean μ and variance σ^2). A $\mathcal{N}(0,1)$ random variable is called *standard Gaussian* (section 2.1).
$(\mathbb{N}_0)_c^{\mathbb{N}}$	Space of multi-indices, $(\mathbb{N}_0)_c^{\mathbb{N}} := \{\, (\alpha \in \mathbb{N}_0)^{\mathbb{N}} \mid$ only finitely many α_i are nonzero $\}$ (section 2.2.3).
$(\mathbb{N}_0)^m_{:\leq p:}$	Set of multi-indices identifying the m-dimensional polynomial chaos of degree p, defined as $(\mathbb{N}_0)^m_{:\leq p:} := \{\alpha \in \mathbb{N}^m \,|\, |\alpha| \leq p\}$ (section 2.2.3).
$(\Omega,\mathcal{B},P)$	A probability space. Ω: set of elementary events; $\mathcal{B}$: the σ-algebra of events; P: probability measure (section 2.1).
ω	ω either specifies an elementary event, $\omega \in \Omega$, or it specifies a sequence of independent random variables $\omega = (\omega_1, \omega_2, \ldots)$
ω_i	$\omega_1, \omega_2, \ldots$ denote independent random variables.
$\boldsymbol{\omega}$	Vector of m independent random variables $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^t$
Ω	Probability space, set of elementary events (section 2.1)
$\phi(x,\gamma)$	Nonlinear transformation used to transform a Gaussian field into a non-Gaussian one, $\kappa(x,\omega) = \phi(x,\gamma(x,\omega))$ (section 3.2.2).
$p_\kappa(k)$	Probability density funciton (PDF) of a random variable κ, defined as $p_\kappa(k) = \frac{dF_\kappa(k)}{dk}$ (section 2.1).
P	Probability measure (section 2.1).
P_κ	Probability measure that is induced by a random variable κ on its range (section 2.1).
$\mathcal{P}_n(G)$	Linear space of polynomials on a Hilbert space G (section 2.2.3).
$\Delta Q_l^{(i)}$	Used in Smolyak construction, $\Delta Q_l^{(i)} := Q_l^{(i)} - Q_{l-1}^{(i)}$, where $Q^{(i)}$ is a quadrature formula (section 7.1).
$Q_Z(\psi)$	High dimensional integration in Z integration points, $Q_Z(\psi) \approx \mathbf{E}\,(\psi(\omega))$ (section 7.1).
R	The spatial region on which a stochastic field or the SPDE is defined, $R \subset \mathbb{R}^d$ (section 3.1).
σ_κ	Standard deviation of the random variable κ, defined as $\sigma_\kappa = \sqrt{\mathrm{var}_\kappa}$ (section 2.1).
(S)	Abstract space of stochastic functions (section 4.1).
$(S)^{\rho,r}$	Hida distribution and test function spaces (section 2.3).
$(S)^{\rho}$	Kondratiev distribution and test function spaces (section 2.3).
$S(\mathbb{R}^d)$	Space of rapidly decreasing functions (section 2.2.2).
$S(\mathbb{R}^d)'$	Space of tempered distributions (section 2.2.2).
S_l^m	Smolyak formula of level l in m dimensions (Eq. (7.1.4)).

T_κ	Operator of stochastic PDE, κ indicates the dependence on the material parameter (section 4.1).
var_κ	Variance of the random variable κ, defined as $\mathrm{var}_\kappa = \mathbf{E}\left(\kappa^2\right) - \mu_\kappa^2$ (section 2.1).
$\Xi_l^{(i)}$	The set of nodes used by the quadrature formula $Q_l^{(i)}$ (section 7.1.4).
x	Point in the spatial domain, $x \in R \subset \mathbb{R}^d$.
y	Point in the spatial domain, $y \in R \subset \mathbb{R}^d$.
Z	The number of integration points (section 7.1).

Part I

Theoretical Aspects of Stochastic Partial Differential Equations

Chapter 2

Basics

To fix notation, the most important basics of probability theory and stochastic analysis are presented condensedly. This chapter is quite technical—the contents are important mainly for the (themselves technical) section 3.1 and chapter 4. The rest of this review may mostly be understood without reading this chapter. Hence, readers not so much interested in the mathematical formalism may skip parts of this chapter with little loss.

2.1 Basics of Probability Theory

First, some basics of probability theory are summarised; see e.g. Bauer (1991); Papoulis (1991); Grigoriu (2002).

A probability space is denoted by $(\Omega, \mathcal{B}, P)$, where Ω is the set of elementary events, $\mathcal{B}$ is the σ-algebra of events and P is the probability measure. The symbol ω specifies an elementary event $\omega \in \Omega$.

Random variables (RVs) are measurable function $\kappa : \Omega \to V$, where V is a measure space, and they will be written in Greek letters. If $V = \mathbb{R}^d$, then $\boldsymbol{\kappa}$ is a *random vector*, which is emphasised by bold Greek letters. The σ-algebra generated by a set of random-variables $\{\kappa_i\}_{i\in\mathcal{I}}$ with an index set $\mathcal{I}$ is called $\Sigma(\{\kappa_i\}_{i\in\mathcal{I}})$.

A random variable κ with values in V induces a probability measure on V called P_κ; P_κ is called the *probability distribution of* κ. The distribution function of a real valued random variable κ is called $F_\kappa(k) = P_\kappa(-\infty, k) = P\{\kappa < k\}$, and—if it exist—its probability density is denoted by $p_\kappa(k) = \frac{dF_\kappa(k)}{dk}$.

RVs are often characterised by their statistics defined as expectation

$$\mathbf{E}\left(g(\kappa)\right) = \int_\Omega g(\kappa(\omega))\, dP(\omega) = \int_{\mathbb{R}} g(\kappa)\, dF_\kappa(\kappa), \tag{2.1.1}$$

where g is some suitable function. Important statistics are the mean $\mu_\kappa = \mathbf{E}\left(\kappa\right)$, the variance $\mathrm{var}_\kappa = \mathbf{E}\left(\kappa^2\right) - \mu_\kappa^2$, the standard deviation $\sigma_\kappa = \sqrt{\mathrm{var}_\kappa}$. The probability

that κ takes values in a P-measurable set may be written as $P\{\kappa \in B\} = \mathbf{E}(\chi_B(\kappa))$, where χ_B is the characteristic function of B.

The covariance is a bivariate statistics $\mathrm{cov}(\kappa_1, \kappa_2) = \mathbf{E}((\kappa_1 - \mu_{\kappa_1})(\kappa_2 - \mu_{\kappa_2}))$ of two random variables κ_1 and κ_2. In general, multivariate statistics may be written for real-valued $\kappa_1, \ldots, \kappa_m$ as

$$\mathbf{E}(g(\kappa_1, \ldots, \kappa_m)) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(k_1, \ldots, k_m)\, dF_{\kappa_1,\ldots,\kappa_m}(k_1, \ldots, k_m), \tag{2.1.2}$$

where g is some function and where $F_{\kappa_1,\ldots,\kappa_m}$ is the joint distribution function of $\kappa_1, \ldots, \kappa_m$.

Let us mention a connection to numerical procedures: When discretising stochastic problems in a Monte Carlo fashion, one usually starts with an abstract probability space $(\Omega, \mathcal{B}, P)$ and then represents the problem in a finite number of independent random variables $\boldsymbol{\kappa}(\omega) = (\kappa_1(\omega), \ldots, \kappa_m(\omega))^T$. One may then work with the probability space induced by $P_{\boldsymbol{\kappa}}$ on the range of the random vector $\boldsymbol{\kappa}$. If the $\kappa_1, \ldots, \kappa_m$ are independent, then Eq. (2.1.2) may computed due to Fubini's theorem as

$$\mathbf{E}(g(\kappa_1, \ldots, \kappa_m)) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(\kappa_1, \ldots, \kappa_m)\, dF_{\kappa_1}(\kappa_1) \cdots dF_{\kappa_m}(\kappa_m). \tag{2.1.3}$$

For such "coordinate systems" of independent RVs, usually vectors of independent Gaussian RVs are used. Reasons are that two Gaussian RVs γ_1, γ_2 are independent if they are uncorrelated, i.e. if $\mathrm{cov}(\gamma_1, \gamma_2) = 0$, and that their linear combinations are also Gaussian. Hence, Gaussian RVs may be transformed to independent RVs by linear algebra.

A Gaussian RV with mean μ and standard deviation σ is denoted by $\gamma = \mathcal{N}(\mu, \sigma^2)$ (we will mark Gaussian RVs by the letter γ). Its probability distribution function is

$$f_\gamma(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{2.1.4}$$

A centred Gaussian random variable with unit variance, $\mathcal{N}(0,1)$, is called *standard Gaussian*. The probability distribution function of a standard Gaussian random variable will be called $\mathrm{erf}(x) := F_{\mathcal{N}(0,1)}(x)$.

Stochastic properties are often specified as nonlinear transformations of Gaussian random variables. For this, one may exploit the well-known fact that a standard Gaussian random variable is mapped to a random variable with distribution function F_κ by the transformation $F_\kappa^{-1}(\mathrm{erf}(\mathcal{N}(0,1)))$ (e.g. Papoulis, 1991).

2.2 Spaces of Random Variables

The mathematical theory of stochastic PDEs involves stochastic analysis. For introductions to this topics see e.g. (Malliavin, 1997; Janson, 1997). Only some basic results relevant for the review are mentioned here.

2.2.1 Gaussian Banach and Hilbert Spaces

The L^p norms for random variables are defined as usual, $\|\kappa\|_p = \mathbf{E}\,(\kappa^p)^{1/p}$ for $0 < p < \infty$ and $\|\kappa\|_\infty = \operatorname{ess\,supp} |\kappa|$. The space $L^p = L^p(\Omega, \mathcal{B}, P)$ is the space of all random variables on $(\Omega, \mathcal{B}, P)$ that satisfy $\|\kappa\|_p < \infty$. Just as in the deterministic case, L^2 is a Hilbert space, and L^p is a Banach space for $1 \leq p \leq \infty$. For $1 \leq p < \infty$, the dual is $(L^p)' = L^q$ with $q^{-1} + p^{-1} = 1$. Furthermore, L^r is a dense subset of L^p whenever $0 < p \leq r \leq \infty$.

For centred variables $\gamma_1, \gamma_2 \in L^2$, the expression $(\gamma_1, \gamma_2)_{L^2} := \operatorname{cov}(\gamma_1, \gamma_2)$ defines a scalar product with norm $\|\gamma\|_2^2 := \operatorname{var}_\gamma$. A *Gaussian Hilbert space* (e.g. Janson, 1997) G is a subspace G of $L^2(\Omega, \mathcal{B}, P)$ that only contains centred Gaussian random variables and that is complete when equipped with the covariance as scalar product. Note that the σ-algebra $\Sigma(G)$ may be smaller than $\mathcal{B}$; this is the usual case when stochastic quantities are approximated in a finite number of independent Gaussian random variables.

An important example for a Gaussian Hilbert space is $(\mathbb{R}^m, \mathcal{B}_m, \Gamma_m)$, where Γ_m is the Gauss-measure. $d\Gamma_m(x) = (2\pi)^{-m/2} \exp(-|x|^2/2)\, dx$, and $\mathcal{B}_m$ is the Borel σ-algebra on $\mathbb{R}^m$. Stochastic properties will be approximated in such spaces in the further review.

2.2.2 Measures on Topological Vector Spaces

Random fields are discussed in chapter 3. They are collections of random variables $\kappa(x)$ indexed by $x \in R \subset \mathbb{R}^d$, but they may also be interpreted as RVs κ with values in a function space V. Let us hence comment on how a probability measure may be constructed on a topological vector space V that will be assumed to be Banach or locally convex. For clearness, first Gaussian measures (see Janson, 1997, example 1.13) are considered before the existence and construction of general measures are discussed.

(a) A Borel probability measure P on V (with the Borel σ-algebra $\mathcal{B}$) is said to be Gaussian, if each continuous linear functional $v' \in V'$, regarded as a random variable on $(V, \mathcal{B}, P)$, is Gaussian. If each $v' \in V'$ is centred Gaussian, the completion of $V' \subseteq L^2(V, \mathcal{B}, P)$ is a Gaussian Hilbert space.

(b) A random variable κ with values in V is said to be Gaussian, if $\omega \mapsto \langle v', \kappa(\omega)\rangle$ is a Gaussian random variable for any $v' \in V'$, where $\langle\cdot,\cdot\rangle$ is the duality paring on $V' \times V$. If κ is centred, then the set of all $\langle v', \kappa(\cdot)\rangle, v' \in V'$, is a linear space isomorphic to the space V' considered in (a).

Probability measures may be specified on the dual V' of a given topological vector space V by their Fourier transform, i.e. by their *characteristic functional*. This is a generalisation of the characteristic function for real valued variables (e.g. Papoulis, 1991) and is defined for $v \in V$ as

$$\Phi_\kappa(v) := \mathbf{E}\left(e^{i\langle\kappa(\cdot),v\rangle}\right) = \int_{V'} \exp\left(i\langle v', v\rangle\right) dP_\kappa(v'). \tag{2.2.1}$$

For a given functional Φ_κ on a nuclear space V, the following theorem states conditions for the existence of a measure P_κ on V'.

2.1 Theorem: *(*Bochner-Minlos, see (Gel'fand and Vilenkin, 1964, Theorem 2, p. 350)*) Any continuous, positive definite functional Φ_κ on a nuclear space V, with $\Phi_\kappa(0) = 1$ is the Fourier transform Eq. (2.2.1) of a countably additive positive normalised measure P_κ on V'.*

For example, the functional $\Phi_\gamma(v) := e^{-1/2\|v\|^2_{L^2}}$ on the space of rapidly decreasing functions $V = S(\mathbb{R}^d)$ satisfies the conditions of the Bochner-Minlos theorem. It thus defines a Gaussian probability measure P_γ on the space of tempered distributions $V' = (S(\mathbb{R}^d))'$ with the Borel σ-algebra $\mathcal{B}$ of $S'(\mathbb{R}^d)$ equipped with the weak-$*$-topology. The measure P_γ is called the *d-parameter white noise measure*, and the corresponding V'-valued random variable γ is called the *d-parameter white noise process* (Hida et al., 1993).

For any $\phi \in S(\mathbb{R}^d)$, a random variable on V' is given by the map $\gamma \mapsto \langle\gamma, \phi\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $V' \times V$. As $S(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$ in the L^2-topology and the map is continuous in this topology, it may be extended to $\phi \in L_2(\mathbb{R}^d)$. For all $\phi_1, \ldots, \phi_k \in L_2(\mathbb{R}^d)$ the random variables $\langle\cdot, \phi_1\rangle, \ldots \langle\cdot, \phi_k\rangle$ defined in this way are jointly Gaussian, and they are independent if the ϕ_i are mutually orthogonal (Hida et al., 1993). The closure of $S(\mathbb{R}^d)$ in $L^2(P)$ is a Gaussian Hilbert space on the probability space $(S(\mathbb{R}^d)', \mathcal{B}, P)$ where $\mathcal{B}$ is the Borel σ-algebra of $S'(\mathbb{R}^d)$ equipped with the weak-$*$-topology (e.g. Janson, 1997, example 1.16).

The requirement in Theorem 2.1 that V is nuclear is important—for example, it is not possible to define a Gaussian measure on an infinite dimensional Hilbert space (e.g. Choquet-Bruhat and Witt-Morette, 1982, chapter VII)—a Gaussian measure may instead be found on a larger topological vector space into which the Hilbert space is densely embedded (e.g. Janson, 1997, example 1.25).

For more examples of spaces with Gaussian probability measure see Hida et al. (1993).

$h_0(x)$	$=$	1	$h_3(x)$	$=$	x^3-3x
$h_1(x)$	$=$	x	$h_4(x)$	$=$	x^4-6x^2+3
$h_2(x)$	$=$	x^2-1	$h_5(x)$	$=$	x^5-10x^3+15x

Table 2.2.1: Hermite polynomials

2.2.3 Polynomial Chaos

The *polynomial chaos* is also called the *Wiener polynomial chaos*, the *Wiener chaos*, or the *Wiener Itô Chaos*. The name may be misleading: the polynomial chaos is a space of orthogonal polynomials, and the name was termed by its inventor Norbert Wiener (Wiener, 1938) long before the modern meaning of the word "chaos" was established.

In the following, let $G \subseteq L^2(\Omega, \mathcal{B}, P)$ be a separable Gaussian Hilbert space. We denote the space of all multivariate polynomials G of degree k by

(2.2.2)
$$\mathcal{P}_k(G) := \{p(\omega_1, \ldots, \omega_m) \mid p \text{ is polynomial of degree } k; \omega_1, \ldots, \omega_m \in G, m < \infty\}$$

and call the space of all polynomials $\mathcal{P}(G) := \cup_{k=0}^{\infty} \mathcal{P}_k(G)$. Denoting by $\bar{\mathcal{P}}_k$ the closure with respect to L^2, we introduce the orthogonal decomposition

(2.2.3) $$\mathcal{H}_{:=0:} := \bar{\mathcal{P}}_0, \qquad \text{(the space of constants)}$$

(2.2.4) $$\mathcal{H}_{:=k:} := \bar{\mathcal{P}}_k \ominus \bar{\mathcal{P}}_{k-1}, \qquad k \in \mathbb{N}.$$

The space $\mathcal{H}_{:=k:}$ is called the *homogeneous chaos of order k*, and $\mathcal{H}_{:\leq k:} := \cup_{l=0}^{k} \mathcal{H}_{:=k:}$ is called the *polynomial chaos of order k*.

The space of polynomials $\mathcal{P}(G)$ is dense in $L^p(\Omega, \sigma(G), P)$ for $0 < p < \infty$ (e.g. Janson, 1997, Theorem 2.11), and the orthogonal decomposition (e.g. Janson, 1997, Theorem 2.6)

(2.2.5) $$L^2(\Omega, \sigma(G), P) = \bigoplus_{k=0}^{\infty} \mathcal{H}_{:=k:}$$

is called the *polynomial chaos decomposition* or the *Wiener chaos decomposition* of $L^2(\Omega, \sigma(G), P)$ (Wiener, 1938).

One may explicitly construct the polynomial chaos by multi-variate Hermite polynomials. These are tensor-products of (uni-variate) Hermite polynomials h_k, where $k \in \mathbb{N}_0$ specifies their degree; see Table 2.2.1. Note that the h_k are orthogonal with respect to the Gaussian measure.

The construction of the multivariate Hermite-polynomials uses *multi-indices*. These are sequences $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ of non-negative integers with only finitely many

$H_{[0,0]}(\omega_1,\omega_2)$	$=$	1	$H_{[1,0]}(\omega_1,\omega_2)$	$=$	$h_1(\omega_1)$
$H_{[0,1]}(\omega_1,\omega_2)$	$=$	$h_1(\omega_2)$	$H_{[1,1]}(\omega_1,\omega_2)$	$=$	$h_1(\omega_1)h_1(\omega_2)$
$H_{[2,0]}(\omega_1,\omega_2)$	$=$	$h_2(\omega_1)$	$H_{[0,2]}(\omega_1,\omega_2)$	$=$	$h_2(\omega_2)$

Table 2.2.2: Two dimensional polynomial chaos of order 2.

non-zero elements, and the set of all multi-indices will be called

$$(\mathbb{N}_0)_c^{\mathbb{N}} := \{\,(\alpha \in \mathbb{N}_0)^{\mathbb{N}} | \text{only finitely many } \alpha_i \text{ are nonzero}\}. \tag{2.2.6}$$

The modulus and factorial of $\alpha \in (N_0)_c^{\mathbb{N}}$ are defined as $|\alpha| = \sum_{i\in\mathbb{N}} \alpha_i$ and as $\alpha! := \prod_{i\in\mathbb{N}}(\alpha_i!)$. Algebraic operations on multi-indices are defined component-wise, i.e. $\alpha+\beta = (\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots)$.

As G was assumed to be separable, it has a countable orthonormal basis of random variables $\omega = \{\omega_i\}_{i\in\mathbb{N}}$. The *multivariate Hermite polynomial* for a multi-index α may be defined as

$$H_\alpha(\omega) := \prod_{i\in\mathbb{N}} h_{\alpha_i}(\omega_i). \tag{2.2.7}$$

This gives an explicit representation of the Wiener chaos: The set of all $H_\alpha(\omega)$ with $|\alpha| = k$ is an orthogonal basis of $\mathcal{H}_{:=k:}$. Hence, because of the decomposition Eq. (2.2.5), any random variable on Ω with finite variance has an L^2 convergent approximation in the multivariate Hermite polynomials; this was shown by Cameron and Martin (1947).

In numerical applications, stochastic quantities may be approximated in the polynomial chaos of order k over a finite dimensional Gaussian Hilbert space G with orthonormal basis $\omega = (\omega_1, \ldots, \omega_m)$. The computations may then be performed using the orthogonal basis of $\mathcal{H}_{:\le k:}$ that consists of all $H_\alpha(\omega)$ with $|\alpha| \le k$, e.g. see Table 2.2.2. Consequently, the m-dimensional polynomial chaos of degree k will be identified by the multi-index set

$$(\mathbb{N}_0)^m_{:\le k:} := \{\alpha \in \mathbb{N}^m \,|\, |\alpha| \le k\}. \tag{2.2.8}$$

For convenience, some properties of the polynomial chaos are collected below:

1. The vector space dimension of the homogeneous chaos in $\dim G = m$ stochastic independent random variables is (e.g. Janson, 1997, Corollary 3.24)

   $$\mathcal{H}^m_{:=k:} = \binom{k+m-1}{m-1}. \tag{2.2.9}$$

   As Table 2.2.3 shows, this number grows rapidly in the number of Gaussian random variables and in the polynomial degree.

Stochastic dimensions m	Polynomial Degree k	Vector space dimensions of polynomial chaos
5	3	56
	5	252
10	3	286
	5	3,003
20	3	1,771
	5	$\approx$ 53,000
100	3	$\approx$ 177,000
	5	$\approx$ 96,000,000

Table 2.2.3: Vector space dimensions of the polynomial chaos of degree k in m independent random variables.

2. The H_α are orthogonal and

$$\mathbf{E}\left(H_\alpha H_\beta\right) = \alpha!\,\delta_{\alpha\beta}, \quad \text{specifically} \quad \|H_\alpha\|_{L^2}^2 = \alpha! \tag{2.2.10}$$

3. As the polynomial chaos is an orthogonal basis of $L^2 := L^2(\Omega, \sigma(G), P)$, any random variable of finite variance $f \in L^2$ has the L^2-convergent expansion

$$f = \sum_\alpha f^{(\alpha)} H_\alpha, \tag{2.2.11}$$

where $f^{(\alpha)} = \frac{1}{\|H_\alpha\|_{L^2}^2}\mathbf{E}\,(fH_\alpha) = (\alpha!)^{-1}\mathbf{E}\,(fH_\alpha)$.

4. The projection $f^{(\alpha)}$ may be computed analytically for smooth random variables. Let $\omega = (\omega_1, \ldots, \omega_m)$ be an orthonormal basis of G and let a random variable $f \in L^2 := L^2(\Omega, \sigma(G), P)$ be given as a function $f(\omega_1, \ldots, \omega_m)$. If all its partial derivatives belong to L^2, then

$$f^{(\alpha)} = (\alpha!)^{-1}\,\mathbf{E}\,(D^\alpha f)\,, \tag{2.2.12}$$

where D^α is the partial derivative with respect to the multi-index (e.g. Malliavin, 1997, Theorem 3.1).

2.3 Stochastic Distributions

Just as generalised functions (distributions) are continuous functionals on test spaces of smooth functions (Gel'fand and Shilov, 1964), generalised random variables (stochastic distributions) are functionals on test spaces of smooth random

variables (Hida et al., 1993; Holden et al., 1996). The spaces of distribution spaces considered here are called the Hida and Kondratiev distribution spaces (Hida et al., 1993; Holden et al., 1996; Benth and Gjerde, 1998).

Analogous to the characterisation of tempered distributions by the Fourier coefficients of their Hermite expansion (e.g. Reed and Simon, 1980, p.143), generalised random variables may be constructed as formal polynomial chaos expansions (Hida et al., 1993; Holden et al., 1996): Let the multivariate Hermite polynomials H_α on a separable Gaussian Hilbert space G be defined as in Eq. (2.2.7). Let V be a separable Hilbert space, let $\rho \in [-1,1]$ and $r \in \mathbb{R}$. For any (formal) expansion $f = \sum_\alpha f^{(\alpha)} H_\alpha$, with $f^{(\alpha)} \in V$ for all multi-indices α, define

$$\|f\|^2_{\rho,r} := \sum_\alpha \|f^{(\alpha)}\|^2_V \,(\alpha!)^{1+\rho}\,(2\mathbb{N})^{r\alpha}, \quad \text{where } (2\mathbb{N})^{r\alpha} := \prod_{j\in\mathbb{N}} (2j)^{r\alpha_j}, \tag{2.3.1}$$

and define $(S)^{\rho,r}$ as the vector space of all such f with $\|f\|_{\rho,r} < \infty$. Then $\|\cdot\|_{\rho,r}$ is a norm, and the spaces $(S)^{\rho,r}$ are separable Hilbert spaces (Hida et al., 1993; Holden et al., 1996) equipped with the scalar product

$$(f,g)_{\rho,r} := \sum_\alpha (f^{(\alpha)}, g^{(\alpha)})_V \,(\alpha!)^{1+\rho}\,(2\mathbb{N})^{r\alpha}. \tag{2.3.2}$$

The dual of $(S)^{\rho,r}$ may be identified with $(S)^{-\rho,-r}$, and the duality pairing on $(S)^{-\rho,-r} \times (S)^{\rho,r}$ is

$$\langle F, f\rangle := \sum_\alpha (F^{(\alpha)}, g^{(\alpha)})_V \,\alpha!. \tag{2.3.3}$$

It is obvious that $(S)^{0,0} = L^2(\Omega)$. For $\rho > 0, r > 0$ the random variables (RVs) $f \in (S)^{\rho,r}$ have coefficients $\|f^{(\alpha)}\|$ that decrease rapidly when the degree $|\alpha|$ grows or when the maximum index i of non-zero elements α_i (the length of α) grows. For $\rho, r > 0$ the spaces therefore contain RVs that have faster decreasing coefficients than required for finite variance. By analogy to the Fourier transform of deterministic functions, one may say that the larger ρ or the larger r, the more regular are the random variables in $(S)^{\rho,r}$. These spaces are test function spaces, similar to the space of rapidly decreasing functions. Their duals $(S)^{-\rho,-r}$ are the spaces of stochastic distributions or of generalised random variables. Members of $(S)^{-\rho,-r}$ are generalised RVs or stochastic distributions, i.e. linear functionals acting on the random test functions.

Holden et. al. Holden et al. (1996) use these spaces to construct (for $\rho \in [0,1]$) the *Kondratiev test spaces* $(S)^\rho := \cap_{k\geq 0}(S)^{\rho,k}$ (with the projective limit topology) and their duals, the *Kondratiev distribution spaces* $(S)^{-\rho} := \cup_{k\geq 0}(S)^{-\rho,-k}$.

We consider now approximations of RVs and generalised RVs in the polynomial chaos: Let $f \in (S)^{\rho,q}$, and let $f^m_{\leq k}$ be its projection onto the m-dimensional

polynomial chaos of degree k, i.e.

$$f^m_{\leq k} := \sum_{\substack{\alpha\in\mathbb{N}_0^m \\ |\alpha|\leq k}} f^{(\alpha)} H_\alpha. \tag{2.3.4}$$

A useful estimate for the error of this approximation was given by Benth and Gjerde (1998):

2.2 Theorem: *(Benth and Gjerde (1998, Theorem 3.1))*
Choose $p > 0$ and $q \in \mathbb{R}$ such that $r := p - q > r^*$, where r^* solves the equation $r^* = 2^{r^*}(r^* - 1)$; note that $r^* \approx 1.53$. Let $\rho \in [-1, 1]$.
Then for any $f \in (S)^{-\rho,-q}$ and $g \in (S)^{\rho,p}$ the inequality

$$\left|\langle f - f^m_{\leq k}, g\rangle\right| \leq \|f\|_{-\rho,-q} \cdot \|g\|_{\rho,p} \cdot c(m,k,p-q) \tag{2.3.5}$$

holds, where

$$c(m,k,r) = \sqrt{c_1(r)m^{1-r} + c_2(r)\Big(\frac{r}{2^r(r-1)}\Big)^{k+1}}, \tag{2.3.6}$$

and where $c_1(r) = (2^r(r-1) - r)^{-1}$ and $c_2(r) = 2^r(r-1)\cdot c_1(r)$.

2.3 Remark: In the proof given in Benth and Gjerde (1998, Theorem 3.1), the estimates depend on $r := p - q$ being greater than r^*. The proof still holds for $p \in \mathbb{R}$. The proof concludes Eq. (2.3.6) from the inequality

$$\|f - f^m_{\leq k}\|_{-\rho,-p} \leq \|f\|_{-\rho,-q} \cdot c(m,k,p-q). \tag{2.3.7}$$

2.4 Remark: This estimate does not give a useful estimate of $\|f - f^m_{\leq k}\|_{L^2}$ in terms of $\|f\|_{L^2}$; stronger regularity assumptions are required to estimate L^2 approximation errors.

2.5 Example: Benth and Gjerde (1998, Example 3.4): If $p = q + 2$ then $r = 2$ and $c(m,k,r)^2 = 1/2(m^{-1} + (1/2)^{k-1})$. Then

$$\|f - f^m_{\leq k}\|_{-\rho,-p} \leq \frac{1}{2}\|f\|_{-\rho,-p+2} \cdot (m^{-1} + 2^{-k+1}). \tag{2.3.8}$$

Chapter 3

Random Fields

Uncertainties in physical quantities varying in time or in space may be modelled by *stochastic processes* or *random fields*. Examples are the price of a stock (temporal randomness), the hydraulic conductivity of soil (spatial randomness), or wind forces acting on a structure (spatio-temporal randomness). Traditionally, the phrase *stochastic process* denotes stochastic uncertainty in time, while the phrase *random field* denotes stochastic uncertainties on a domain in higher dimensions. Nonetheless, the mathematical definitions in the literature are often the same. As we will mostly be concerned with randomness in space, we use the term *random fields*.

Mathematically oriented introductions to stochastic processes are given e.g. by Bauer (1991), Doob (1953), Krée and Soize (1986), or briefly in Øksendal (1998) and Kloeden and Platen (1995). More practically oriented textbooks are e.g. Papoulis (1991), Grigoriu (1995, 2002), or Van Trees (1968). For a mathematically oriented work on random fields see Adler (1981), or Vanmarcke (1988) for a more practically oriented exposition. For a comprehensive treatment of random fields with applications to the earth sciences see Christakos (1992). For introductions to generalised stochastic processes and random fields see e.g. Gel'fand and Vilenkin (1964), Krée and Soize (1986), Christakos (1992), or Holden et al. (1996).

3.1 Definitions of Random Fields

Now, definitions and interpretations of random fields are presented in a somewhat informal manner. The specification of random fields in practice is discussed afterwards.

3.1.1 Characterisation of Random Fields

A random field κ on a region $R \subset \mathbb{R}^d$ and on a probability space $(\Omega, \mathcal{B}, P)$ may be interpreted as a set of random variables indexed by $x \in R$ or as a function-valued random variable. In both interpretations, a random field is a measurable mapping

$$\kappa : R \times \Omega \longrightarrow \mathbb{R}.$$

Recall that any random variable corresponds to a probability space that has as elementary events the range and as probability measure the probability distribution of the random variable; see section 2.1. For random fields, this gives the following two characterisations (Adler, 1981; Øksendal, 1998; Christakos, 1992):

Probabilistic Characterisation: A random field κ is a set of random variables

$$\kappa(x) := \kappa(x, \cdot) : \Omega \longrightarrow \mathbb{R} \tag{3.1.1}$$

indexed by $x \in R$. It may be defined by specifying all finite dimensional ("fi-di") distribution functions $F_{x_1 \ldots x_n}(\hat{x}_1, \ldots, \hat{x}_n) = P\{\kappa(x_1) \leq \hat{x}_1 \wedge \ldots \wedge \kappa(x_n) \leq \hat{x}_n\}$, with $x_1, \ldots, x_n \in R$ and $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}$. It will be assumed here that κ is separable, a condition which e.g. guarantees that certain properties of realisations—like continuity—are determined by the fi-di distributions (Doob, 1953; Adler, 1981).

The probability space needs not to be specified explicitly, as it may be constructed from the fi-di distributions under weak consistency conditions (e.g. Øksendal, 1998, Theorem 2.1.5). As discussed below, Ω may be identified with the set of realisations, and the fi-di distributions implicitly define a probability measure on the space of realisations.

Measure Theoretic Characterisation: Alternatively, a random field κ may be defined as a random variable that has as values functions on a region $R \subset \mathbb{R}^d$. Any elementary event ω yields a *realisation*

$$\kappa(\cdot, \omega) : R \longrightarrow \mathbb{R}.$$

The realisations may be identified with the elementary events,

$$\text{“}\omega(x) \equiv \kappa(x, \omega)''\text{,} \tag{3.1.2}$$

and then one may identify Ω with a subset $\Omega \subset \{\omega \,|\, \omega : R \to \mathbb{R}\}$. Defining κ amounts to specifying a probability measure P_κ on this function space Ω. This may in principle be done in conformance with given fi-di probability distributions (e.g. see Øksendal, 1998). But the interaction of the topological structure with the measure-space structure complicates this as it may not be possible to define

the measure on the Borel-σ-algebra; for example, one cannot define a Gaussian measure on an infinite-dimensional Hilbert-space (e.g. Janson, 1997, example 1.25). For nuclear spaces, the measure on the topological space may be defined using the Bochner-Minlos (see section 2.2.2).

3.1.2 Generalised Random Fields

Highly erratic random fields, like white noise, cannot be described as above. Instead, generalised random fields are required, see e.g. Gel'fand and Vilenkin (1964); Krée and Soize (1986). For example, responses of elastic structures under white noise wind loads may by generalised random fields (Walsh, 1984; Krée and Soize, 1986).

Generalised Random Fields: In the view presented by Gel'fand and Vilenkin (1964) or Krée and Soize (1986), a generalised random field (GRF) κ is a random variable, which has as realisations generalised functions. As in the measure theoretic characterisation above, the probability space may be identified with the space of realisations. Realisations are chosen as tempered distributions, $\Omega = S(\mathbb{R}^d)'$, where $S(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing functions with the weak-$*$-topology. The random events $\mathcal{B}$ are the Borel sets of Ω. Other spaces of functionals may also be used (e.g. see Christakos, 1992; Hida et al., 1993). The definition of generalised random fields by their characteristic functional via the Bochner-Minlos theorem and the construction of the white noise generalised random field has been discussed in section 2.2.2.

Fields of Kondratiev Distributions: In stochastic partial differential equations, multiplicative noise may occur. It is then necessary to define the product of generalised random fields. This is difficult, as it is not obvious how one may define the product of two continuous linear functionals so that the result is again a continuous linear functional.

Holden et al. (1996) overcome the problems of multiplying generalised random fields by defining generalised random fields as Kondratiev distributions (see section 2.3) and interpreting products as Wick products. This is discussed in section 4.2.2.

3.2 Specifying Random Fields

Abstract definitions of random fields have been given above. For engineering applications, the definition of a random field by all its finite dimensional distributions or by a measure on a probability space is not practical.

Some models of Gaussian and non-Gaussian random fields are discussed next. For more in-depth discussions see e.g. the text book by Grigoriu (1995) or the comprehensive article (Grigoriu, 1997).

3.2.1 Gaussian Random Fields

A Gaussian random field $\gamma(x,\omega)$ on a region $R \subset \mathbb{R}^d$ is a random field, for which all finite dimensional distributions are jointly Gaussian.

Gaussian random fields are probably the most frequently used models. On the one hand, this is due to theoretical reasons: they occur naturally because of the central limit theorem, and they are the maximum entropy model if only second-order information are available (see section 3.4.4). On the other hand, they are easy to work with: they are defined by their second-order statistics, uncorrelated Gaussian RVs are independent, and linear combinations of Gaussian RVs are Gaussian.

The specification of a Gaussian random field is simple: any finite number of Gaussian RVs are completely determined by their joint second-order statistics (e.g. Papoulis, 1991). Hence, any Gaussian random field γ is also determined completely by its second-order statistics, i.e. by its mean $\mu_\gamma(x) = \mathbf{E}(\gamma(x,\omega))$ and by its covariance function $\mathrm{cov}_\gamma(x,y) = \mathbf{E}\left((\gamma(x,\omega)-\mu_\gamma(x))(\gamma(y,\omega)-\mu_\gamma(y))\right)$ for $x,y \in R$.

However, not every function $\mathrm{cov}_\gamma(x,y)$ is a valid covariance function. For this, cov_γ needs to be symmetric and positive semi-definite (e.g. Christakos, 1992, Ch. 3.1), a property that is not always easy to assert, especially in higher spatial dimensions. Some criteria are given in section 3.3, and some common covariance functions are presented in section 3.4.

As discussed in section 3.1, the finite dimensional distributions implicitly define the probability space. Hence, the knowledge of the second-order statistics is sufficient to define a Gaussian random field. Moreover, for any given (valid) covariance function $\mathrm{cov}_\gamma(x,y)$ and mean function $\mu_\gamma(x)$, there is a Gaussian random field γ on R having these as second-order statistics.

3.2.2 Specifying Non-Gaussian Random Fields

As section 3.4 will show, the modelling of non-Gaussian random fields is still an active area of research. Some models for non-Gaussian fields commonly used are discussed below. Some textbooks treating this topic are Deutsch (1962), Grigoriu (1995), Prigarin (2001), Ogorodnikov and Prigarin (1996).

Often, only second-order statistics and the marginal distribution of a random field are prescribed (there is seldom enough statistical information available to specify higher order statistics).

Transformations of Gaussian Processes: It is a well-known fact that a standard distributed Gaussian random variable $\mathcal{N}(0,1)$ can be mapped to a random variable with distribution function F_κ by the transformation $F_\kappa^{-1}(\mathrm{erf}(\mathcal{N}(0,1)))$, where erf is the Gaussian distribution function, e.g. see Papoulis (1991) and section 2.1.

Exploiting this fact, a non-Gaussian Random field may be specified as a non-linear transformation of a Gaussian random field $\gamma : R \times \Omega \longrightarrow \mathbb{R}$ with mean $\mu_\gamma(x)$ and covariance function $\mathrm{cov}_\gamma(x,y)$. Without restriction of generality, assume that it is centred, $\mu_\gamma(x) = 0$, and that it has unit variance $\mathrm{var}_\gamma(x) = \mathrm{cov}_\gamma(x,x) = 1$.

Using a nonlinear transformation

$$\kappa(x,\omega) = \phi(x,\gamma(x,\omega)) := F^{-1}_{\kappa(x)} \circ \mathrm{erf}(\gamma(x,\omega)), \tag{3.2.1}$$

a non-Gaussian random field κ is defined, which has at any point x the marginal distribution $F_\kappa(x)$. If ϕ is the inverse of a distribution function without atoms, it is monotonic in the second argument and hence invertible. This representation is sometimes called the "method of inverse distribution functions" (Ogorodnikov and Prigarin, 1996).

The p-th order moments of κ are

$$\mathbf{E}\left(\kappa(x,\omega)^p\right) = \int_{\mathbb{R}} \phi(x,w)^p dF_\gamma(w), \tag{3.2.2}$$

where dF_γ is the standard Gaussian measure from Eq. (2.1.4) (with $\mu = 0, \sigma = 1$). The covariance of κ is

$$\mathrm{cov}_\kappa(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x,w_1)\phi(y,w_2) dF_{\gamma(x),\gamma(y)}(w_1,w_2) - \mu_\kappa(x)\mu_\kappa(y), \tag{3.2.3}$$

where $dF_{\gamma(x),\gamma(y)}$ is the joint probability density of the two Gaussian random variables $\gamma(x)$ and $\gamma(y)$.

Usually, not the second-order statistics of γ are prescribed, but those of κ. Hence, the combination of ϕ and $\mathrm{cov}_\gamma(x,y)$ has to be selected so that $\kappa(x,\omega)$ satisfies given second-order statistics $\mu_\kappa(x)$ and $\mathrm{cov}_\kappa(x,y)$.

If $\phi(x,\gamma(x,\omega)) = \phi(\gamma(x,\omega))$ and if γ is homogeneous (see section 3.3.1), then κ is called a translation random field. For such fields and for $R \subset \mathbb{R}$, methods for obtaining cov_γ from cov_κ are given by Grigoriu (1995, 1998): in this case, the relation between cov_κ and cov_γ may be inverted if the distribution F has no atoms, and the relation may be obtained by evaluating Eq. (3.2.3) analytically or numerically, or by obtaining a differential equation for cov_κ depending on cov_γ (Deutsch, 1962). Some analytical formulas for various marginal distributions including the uniform, the lognormal, the arcsin, the Rayleigh, and the exponential distribution, are given by Ogorodnikov and Prigarin (1996) and by Grigoriu (1995).

However, arbitrary choices of marginal distribution and target covariance cov_κ may lead to inconsistencies. It is further possible that the resulting cov_γ is not

non-negative definite and hence not a covariance function. This problem and sufficient and necessary conditions for the existence of a random process Eq. (3.2.1) with given marginal distribution and correlation structure are discussed by Ogorodnikov and Prigarin (1996) and by Grigoriu (1995).

Another procedure for finding cov_γ for regions R in higher dimensions may be found in Sakamoto and Ghanem (2002). There, the non-Gaussian process is expanded as $\kappa(x) = \sum_i \kappa_i(x) h_i(\gamma(x))$ where h_i denote the Hermite polynomials, and a set of nonlinear equations is obtained for the correlation of the Gaussian process γ.

Representation as Expansions It is sometimes proposed to represent a non-Gaussian random field as a finite sum of independent centred (non-Gaussian) random variables $\kappa_i(\omega)$ times deterministic functions $k_i(x) : R \longrightarrow \mathbb{R}$ as

$$\kappa(x,\omega) = \overline{\kappa}(x) + \sum_{i=1}^{M} k_i(x)\,\kappa_i(\omega). \tag{3.2.4}$$

This model is used e.g. by Deb et al. (2001); Babuška and Chatzipantelidis (2002) and by Lucor and Karniadakis (2003). Representations of this form may be obtained from experimental data by a principal component analysis (Babuška et al., 2002a); see section 5.2.

Similar representations with non independent random variables κ_i are used in statistical inference, $\kappa(x,\omega) = \overline{\kappa}(x) + \sum_{i=1}^{M} k_i(x)\hat{\kappa}(x_i,\omega)$, where the $\hat{\kappa}_i(x_i,\omega)$ are measurements at some position x_i and where the $k_i(x)$ are weighting functions (Ripley, 1988).

The advantage of this representation is that the field is described in a finite number random variables and hence the discretisation techniques of chapter 5 need not to be applied.

However, if the κ_i are non-Gaussian (nor α-stable, see below), the marginal distribution of κ is usually not known analytically. For example, some of the papers cited above use uniformly distributed κ_i and do not address the question of what marginal distribution the resulting field will have.

One way to prescribe second-order statistics for κ may be to define κ as the first terms of the Karhunen–Loève expansion (see section 5.1.2) of a random field with given covariance. This yields a series where the κ_i are uncorrelated but not independent. If then the higher order correlations between the random variables κ_i are neglected, a good approximation of the desired field may be obtained, but the author is not aware of studies on this.

α-stable random variables: These are random variables defined for $\alpha \in (0,2]$. They are Gaussian for $\alpha = 2$ and have non-Gaussian distributions with heavy tails

for $\alpha \in (0,2)$. α-stable RVs are mentioned as they may be used to numerically model random fields with infinite variance, but they will not be discussed in detail; for introductions see the textbook by Samorodnitsky and Taqqu (1994), the collection by Cambanis et al. (1991), or the brief introduction by Grigoriu (1995).

The class of α-stable random variables is characterised by the property that linear combinations of α-stable random variables are again α-stable random variables. A generalised central limit theorem holds for α-stable random variables: the family of α-stable distributions contains all limiting distributions of sums of i.i.d. random variables. This simplifies the handling of series expansions with α-stable random variables as coefficients as the resulting marginal distributions are known.

Near the origin the probability distributions of α-stable RVs behave similarily to the Gaussian distribution, but their tails are the heavier (i.e. more slowly decreasing), the smaller α is. The parameter $\alpha \in (0,2]$ determines the stochastic regularity of the random variables: for $\alpha = 2$, Gaussian random variables are included. For $\alpha < 2$ the α-stable random variables have only moments of order less than α and hence have infinite variance.

Other Models: Other models for non-Gaussian random fields not discussed here include transformations with memory, non-Gaussian autoregressive moving averages (ARMA) with α-stable noise, filtered Poisson processes, parametric random field models, or models based on stochastic differential equations; e.g. see Grigoriu (1995).

3.3 Properties of Random Fields

The properties of a random field should match these of the physical quantity being modelled—on the one hand with respect to statistical properties of the ensemble, like type of distribution function, homogeneity or isotropy of the field—on the other hand with respect to properties of realisations as continuity, differentiability, positivity (i.e. questions about the space of realisations Ω). Difficulties in achieving this partly stem from the non-obvious ways in which the probability measure and the functional-analytic properties of sample functions interrelate.

3.3.1 Ensemble Properties

The following properties are common assumptions in modelling random fields. For example, it is common in the earth sciences (e.g. Christakos, 1992, p.56) to assume that random fields observed in nature are either homogeneous and isotropic

on the domain of interest, or that they are random fields with homogeneous increments.

Homogeneity: A random field κ is strictly homogeneous (or "stationary" in 1D) if its finite dimensional distributions are invariant under translations. A strictly homogeneous random field is (weakly) homogeneous, i.e. it has constant mean, and its covariance is a function of separation distance only, $\mathrm{cov}_\kappa(x,y) = c(x-y)$. Every homogeneous Gaussian random field is strictly homogeneous (e.g. Adler, 1981), but the converse does not hold.

Homogeneous random fields may be defined by their spectral representation. This provides a practical way of defining valid covariance functions: $\mathrm{cov}_\kappa(x,y)$ is a covariance function, if it is non-negative definite. This is in general difficult to check (see e.g. Gel'fand and Vilenkin, 1964, Chapter II), but for homogeneous random fields, Bochner's theorem (e.g. Reed and Simon, 1975, Theorem IX.9) may be employed. It states that a continuous function $c(h), h \in \mathbb{R}^d$ is non-negative definite if and only if $c(h) = \int_{\mathbb{R}^d} \exp\big(ir \cdot h\big) dQ(r)$, where $Q(r)$ is a non-negative finite measure. Hence, if $c(h)$ is the Fourier transform of a bounded non-negative function, it is non-negative definite and if it is also symmetric, then $\mathrm{cov}_\kappa(x,y) := c(x-y)$ is a covariance function; cf. also the Bochner-Minlos Theorem 2.1.

Isotropy: A homogeneous random field is isotropic if its fi-di distributions are invariant under orthogonal transformations; it is weakly isotropic if its covariance is a function of the absolute value of separation distance only, $\mathrm{cov}_\kappa(x,y) = c(\|x-y\|)$. Any weakly isotropic Gaussian field is strongly isotropic.

Valid isotropic correlation functions and a criterion for checking whether a function is a valid covariance function for an isotropic field κ on $\mathbb{R}^d$ are given by Christakos (1992, Section 2.8.3).

Sufficient conditions for $C(x) := c(\|x\|)$ for $x \in \mathbb{R}^d, d = 1,2,3$ to be non-negative definite are that $c'(0) < 0$, that $\lim_{r\to\infty} c(r)/r^{(1-d)/2} = 0$, and that $\forall r \geq 0$

$$\begin{array}{lll} & \text{for } d=1: & c''(r) \geq 0 \\ (3.3.1) & \text{for } d=2: & \int_r^\infty u(u^2-r^2)^{-1/2} dc''(u) \geq 0 \\ & \text{for } d=3: & c''(r) - r\,c^{(3)}(r) \geq 0. \end{array}$$

For example, the exponential model $c(r) = \exp(-r/a)$ satisfies these conditions for $d = 1,2,3$ and hence is an admissible covariance model for isotropic random fields in one, two, and three dimensions. More valid correlation functions are given by Christakos (1992).

3.3.2 Properties of Realisations

The question about properties of realisations is equivalent to the question on how the probability space Ω is defined in the measure theoretic construction of section 3.1. For instance, if realisations are continuous, then $\Omega \subset C(R)$ with $R \subset \mathbb{R}^d$.

Two different views prevail in the literature: theoretical texts often construct Ω and the measure explicitly, while application oriented works specify its fi-di distributions and do not characterise the space of realisations explicitly.

If the random field is given by its fi-di distribution functions, it is difficult to make almost sure statements about sample function regularity and hence about Ω (some results are in Adler, 1981; Walsh, 1984), but for practical considerations it is often sufficient to consider regularity in a mean square sense (e.g. see Christakos, 1992; Adler, 1981; Vanmarcke, 1988).

Continuity: A random field κ is mean square continuous (m.s.-continuous) in x if $E(\|\kappa(x_n) - \kappa(x)\|^2) \to 0$ for $x_n \to x$. If κ has zero mean, then this is the case if and only if its covariance function $\mathrm{cov}_\kappa(x,y)$ is continuous in $x = y$ (Adler, 1981).

Almost sure continuity of sample functions is a stronger property, and a sufficient condition (Adler, 1981) is that constants $c_1 > 0, c_2 > 0, c_3 > c_1$ exist with

$$\mathbf{E}\left(|\kappa(x+h) - \kappa(x)|^{c_1}\right) \leq \frac{c_2 |h|^{2d}}{|\log|h||^{1+c_3}}. \tag{3.3.2}$$

For $c_1 = 2$ this yields a sufficient condition for almost sure continuity based on the covariance function.

Differentiability: The partial mean square derivatives (m.s.-derivatives) of a zero-mean random field exist if the covariance function is twice differentiable. Then the mean square partial derivatives $\partial_{x_i}\kappa(x) := l.i.m._{\varepsilon \longrightarrow 0} \frac{\kappa(x+\varepsilon e_i) - \kappa(x)}{\varepsilon}$ exist and are random fields with covariance $\mathrm{cov}_{\partial_{x_i}\kappa}(x,y) = \frac{\partial^2 \mathrm{cov}_\kappa(x,y)}{\partial x_i \partial y_i}$ (Adler, 1981). Their cross covariances with the original field are $\mathrm{cov}_{\partial_{x_i}\kappa,\kappa}(x,y) = \frac{\partial \mathrm{cov}_\kappa(x,y)}{x_i}$, and if κ is Gaussian, then its m.s.-derivatives are Gaussian, too. Extension to higher order derivatives is straightforward.

As for almost sure continuity, almost sure differentiability of realisations is difficult to characterise. If Eq. (3.3.2) is satisfied for the mean square partial derivatives of $\kappa(x,\omega)$, then the process is a.s. differentiable with continuous partial derivatives (Christakos, 1992).

Physical Validity of Distribution: Often, physical quantities are bounded or positive. The random field should reflect that. One practical way to achieve this

is to use a nonlinear transformation of a Gaussian field as in section 3.2.2, where the transformation ϕ in Eq. (3.2.1) is positive and bounded.

Many publications on stochastic mechanics use Gaussian or lognormal random fields as material parameters, e.g. for modelling the elastic modulus of a structure. This approach is doubtful as every Gaussian variable has a positive probability of being negative. If such models are used as materials, ill-posed problems may result; see section 4.3. Similarly, the case of unbounded material parameters needs to be considered carefully, as the resulting operator may then be unbounded; see section 4.3.

3.4 Models for Random Fields

The main line of work in stochastic mechanics has so far been on methods for solving systems with given random fields, and hence publications often choose models for random fields (correlation structure, marginal distributions) without justifying the choice by experimental evidence. This section discusses how models for random field may be chosen in applications.

3.4.1 Common Correlation Models

The covariance of a homogeneous field κ is often modelled (e.g. Christakos, 1992) as

$$\mathrm{cov}_\kappa(h) = c(h^t \boldsymbol{G} h), \qquad \text{or as} \tag{3.4.1}$$

$$\mathrm{cov}_\kappa(h) = c(\sqrt{h^t \boldsymbol{G} h}), \qquad h \in \mathbb{R}^d, \tag{3.4.2}$$

where $\boldsymbol{G}$ is a non-negative matrix whose eigenvectors denote the directions of anisotropies. If $\boldsymbol{G}$ is the identity matrix, then an isotropic field is obtained.

In most publications on stochastic mechanics, the one dimensional correlation function c is chosen as one of the correlation models presented below—the models presented here describe the output of a linear system excited by a Gaussian white noise process (Vanmarcke, 1988; Ogorodnikov and Prigarin, 1996):

Autoregressive Correlation Models: The exponential covariance function

$$c_a(r) = \sigma^2 \exp\left(-a^{-1}|r|\right), \quad r \in \mathbb{R} \tag{3.4.3}$$

is obtained as the covariance of the first order autoregression for a discrete series of random variables

$$\kappa_t = c\kappa_{t-1} + w_t, \tag{3.4.4}$$

where κ_t is the random variable at $t \in \mathbb{Z}$ and w_t is an i.i.d. random series. For the continuous case it corresponds to a random process κ satisfying the Langevin equation of Brownian motion $\dot{\kappa}(t) + a^{-1}\kappa(t) = w(t)$, where $w(t)$ is white noise with intensity $G_0 = 2/(\alpha\pi)$ (Vanmarcke, 1988). The resulting process is Markovian and mean square continuous but not differentiable. Its correlation radius is a.

If a random field on $R \subset \mathbb{R}^d$ is defined for $d = 1,2,3$ as in Eq. (3.4.2) or Eq. (3.4.1) with this one dimensional correlation model, the result is an admissible covariance function (Christakos, 1992). Such covariance structures are very common in publications on stochastic finite elements, e.g. in Ghanem and Spanos (1991b) for the bending rigidity of a beam, in Xiu and Karniadakis (2002a) for Gaussian random velocities in a channel flow, in Jardak et al. (2002) for stochastic transport velocities of an advection equation. A similar model is used by Ghanem (1998b), who models the permeability of sand by a covariance function of the form $\mathrm{cov}_\kappa(x,y) = \prod_{i=1}^{d} c_{a_i}(|x_i - y_i|)$.

The autoregression model Eq. (3.4.4) is unilateral: the dependence of κ_t on its neighbours is only in one direction, and such a model is plausible for processes in time. Xiu and Karniadakis (2002b) note that for processes in space a bilateral autoregression $\kappa_t = c_1\kappa_{t-1} + c_2\kappa_{t+1} + w_t$ would be more realistic. They refer to Whittle (1954) who notes that it is unnecessary to use this bilateral model in one dimension, as it may there be reduced to the unilateral model.

However, for random fields in more then one dimension Xiu and Karniadakis (2002b) call the use of this correlation model into question. They refer to Whittle (1954) who showed that in two dimensions the correlation $C(r) = \exp(-r/a)$ (where r is the distance between two points) corresponds to the physical system

$$(\Delta - a^{-2})^{3/4}\kappa(x,y) = w(x,y), \tag{3.4.5}$$

where Δ is the Laplace operator. As the physical mechanism leading to such a relation is difficult to visualise, Whittle (1954) suggests the use of the autoregressive model

$$\kappa_{st} = c(\kappa_{s+1,t} + \kappa_{s-1,t} + \kappa_{s,t+1} + \kappa_{s,t-1}) + w_{st} \tag{3.4.6}$$

for a discrete process on a grid, which corresponds in the continuous case to a Laplace equation with white noise forcing

$$(\Delta - b^{-2})\kappa(x,y) = w(x,y), \tag{3.4.7}$$

where b is an appropriate constant. The resulting covariance in two dimensions is then with an appropriate a

$$C(r) = \frac{r}{a}K_1\Big(\frac{r}{a}\Big), \tag{3.4.8}$$

where K_1 is the modified Bessel function of the second kind with order 1.

Smooth Model: The "Gaussian" type covariance

$$c_a(r) = \sigma^2 \exp\left(-a^{-2} r^2\right), \qquad r \in \mathbb{R} \tag{3.4.9}$$

is another very frequently used model. Its correlation radius is $r_c = \frac{\sqrt{\pi}a}{2}$. The associated process is smooth: it is m.s. continuous and differentiable of any order.

If this covariance is used in Eq. (3.4.2) in $\mathbb{R}^d$, a valid covariance model is obtained for arbitrary d as can directly be seen from Bochner's theorem, see section 3.3.1.

3.4.2 Statistics from Sampling

The statistical information about random fields modelling a real world phenomenom may be obtained by measurements. For overviews on the sampling of random fields see e.g. (Ripley, 1988), Christakos (1992), Sobczyk and Kirner (2001), Martinez and Martinez (2002), or Smith (2001).

Obtaining statistical information for spatial random fields is difficult as the standard methods of statistics or time series analysis cannot be directly applied to spatial data. For example, there may be different trends in different directions, and data sets are usually taken on a bounded subset of the $\mathbb{R}^d$. For points on the borders of this domain, edge corrections have to be applied (Ripley, 1988).

The fi-di distributions required for the definition of a random field comprise an infinite number of parameters, hence a valid probabilistic model of a physical quantity cannot be obtained from experiments alone and measurements have to be combined with additional hypotheses.

Such hypotheses are presented by Christakos (1992, chapter 7), where it is argued that the mathematical modelling of natural processes by random models requires certain fundamental non testable working hypotheses, auxiliary hypotheses, and a heuristic for determining spatial correlation models. The fundamental hypotheses proposed there require that any real world observation is a realisation of a random field, that all calculations may be based on second-order information, and that one realisation of the natural process contains all statistical information about the ensemble. The auxiliary hypotheses for the random fields include e.g. homogeneity and isotropy.

The validity of these hypotheses for real-world problems may be put into question, but they are a compromise between the requirement to provide a theoretical basis for the description of random fields occurring in nature and common practice. For example, in common practice soil properties are usually characterised by second moment information (given by a variogram, see below), and spatial properties need to be assumed to be homogeneous or isotropic, as there often is only a small number of data samples (Ripley, 1988). However, the isotropy assumption

is problematic; for example, the samples need not to be isotropic, even when the underlying spatial process is (Ripley, 1988).

In the earth sciences, statistics for homogeneous random fields are often given as semivariograms (e.g. Smith, 2001; Christakos, 1992). A semivariogram $v(h)$ measures the variance of the difference between samples at different places depending on the lag vector $h \in \mathbb{R}^d$,

$$v(h) = \mathrm{var}(\kappa(x) - \kappa(x+h)). \tag{3.4.10}$$

The semivariogram is similar to the covariance, $\mathrm{cov}_\kappa(x, x+h) = \mathrm{var}_\kappa - \frac{1}{2}v(h)$, but it is subject to smaller errors as it does not depend on the measured mean (Christakos, 1992). Once a semivariogram has been measured it needs to be checked whether or not it is admissible, i.e. whether it defines a valid covariance; see Christakos (1992) for criteria.

According to Eq. (3.3.2), the behaviour of the variogram at the origin determines the regularity of the random field. However, the regularity may depend on the scale used in the sampling procedure. Often, the extrapolation to a zero lag yields $v(0) \neq 0$. This so-called *nugget effect* may result either from incorrect sampling or from the existence of discontinuities (nuggets) in the material (Smith, 2001). If nugget effects occur, the resulting random field is irregular and it may be necessary to model it as a generalised random field; see section 3.1.2.

For special cases of homogeneous fields, statistics can be computed from one sample even without the assumption of ergodicity (Babuška et al., 2002a)—this is discussed here in one dimension: Assume that Z independent observations $\kappa_1(x), \ldots, \kappa_Z(x), x \in \mathbb{R}$, of a homogeneous random field κ with covariance $\mathrm{cov}_\kappa(x,y) = c(x-y)$ are available at sampling points $x_1, \ldots, x_N$. Then an unbiased estimator for the mean is $\mu_{ZN} = \frac{1}{ZN}\sum_{i=1}^{Z}\sum_{j=1}^{N}\kappa_i(x_j)$. This estimator is consistent (i.e. converges to the true mean for $N, Z \longrightarrow \infty$) for a constant number of realisations Z if and only if $\int_0^1 (1-x)c(x)\,dx = 0$ (Babuška et al., 2002a). Hence, homogeneity is sufficient for obtaining a valid estimate of the mean only for special types of covariance functions. This problem may be overcome in practice (Babuška et al., 2002a)—see there for a discussion of similar problems in estimating the covariance of a homogeneous field from a small number of samples.

If statistics have been obtained for some kind of problem, the data measured for an actual observation may be incorporated by preconditioning the stochastic fields—see Hoshiya and Yoshida (1997) and the references therein. A methodology for this is Kriging (Matheron, 1963), which is an optimal prediction of data at some point using linear combinations of nearby observations. An introduction is given e.g. by Sobczyk and Kirner (2001).

3.4.3 Connection to Homogenisation

Real world measurements are often difficult to obtain. To obtain a model for the random fields, realisations of the micro structure of random materials (e.g. see Sobczyk and Kirner, 2001; Torquato, 2000) may be generated and statistics may be obtained from these. The assessment of their statistical properties may then be based on a random meso-scale description of the material obtained on a representative volume element (RVE), e.g. by homogenisation techniques (Ostoja-Starzewski, 1994). It has been noted that most models of random fields lack a connection to the material microstructures (Ostoja-Starzewski, 2001). A recent overview on upscaling and homogenisation has been given in the collection (Dormieux and Ulm, 2002). For overviews on the effects of micro-scale fluctuations of heterogeneous materials see e.g. Zohdi and Wriggers (2001) and the references therein.

Huyse and Maes (1999, 2000, 2001) obtain random microstructures describing an elastic continuum by performing a Delauney triangulation of points distributed according to a homogeneous Poisson random field (e.g. see Sobczyk and Kirner, 2001, Chapter 3.2). They interpret the links in the triangulation as springs with deterministic stiffness. A homogenisation of the resulting structure yields for each realisation of the Poisson process an elastic stiffness tensor, and a Monte Carlo procedure is used to obtain the statistical properties of the elastic modulus. They conclude that the usual constitutive laws may not be simply randomised as this leads to a perfect correlation between the axial shear modes while stochastic homogenisation results in no correlation at all. It is further concluded that modelling only Young's modulus as a random field usually underestimates the true variability of the random field.

Another approach to finding the probabilistic distribution of effective parameters is presented by Kamiński and Kleiber (2000) and Kamiński (2001). There, a periodic composite structure is considered, where the material is defined as a random field. Due to the random field, the homogenisation problem for the reference volume element (RVE) becomes a stochastic equation, which is solved by the second-order perturbation method (e.g. Kleiber and Hien, 1992) to obtain stochastic effective material parameters.

3.4.4 Choice of the Marginal Distribution

If one accepts the hypothesis that distributions of the random field should be chosen so that existing information is exploited as much as possible, then the distributions should be based on the *principle of maximum entropy* (Jaynes 1957, Shannon 1948), (Papoulis, 1991, e.g.). Only considering marginal distributions, it may be concluded that (Papoulis, 1991)

- if it is only known that a variable takes value in a bounded region, it should be modelled by a uniform probability distribution.
- if the available information are the second-order statistics, then a Gaussian distribution should be chosen.
- if the information available are the higher order moments of the property and if a (possibly unbounded) interval $[a,b]$, $a,b \in [-\infty,\infty]$ is known in which the values lie, where $[a,b] = \mathbb{R}$ if no information on the range is available, then the maximum entropy density is $f(x) = c\chi_{[a,b]}(x)\exp\left(-\sum_{i=1}^{k}\lambda_i x^i\right)$. Here, the constants $c,\lambda_1,\ldots,\lambda_k$ are the solutions of a system of nonlinear equations involving the moments.

3.5 Conclusions

In order to obtain stochastic models for random fields, their statistical properties have to be measured or obtained by other means. Most publications of stochastic finite element techniques do not employ real life statistical models. Instead, they use the exponential model Eq. (3.4.3) or the smooth model Eq. (3.4.9) for the covariance structure. The reason for this may be that obtaining spatial statistical data is difficult (Ripley, 1988), but as the discussion in section 3.4.1 shows, the exponential model may not be the right choice in two or three spatial dimensions.

The quality of statistical information required for a good model for a random field depends on the intended application. For example, Elishakoff (1999b) argues that in the prediction of small failure probabilities, small changes in the probability distribution function (PDF) of material parameters may have a significant effect on the computed system's reliability—even if all moments of the chosen PDF match the moments of the true PDF. However, as will be seen in section 4.3 for elliptic problems, the system's answer depends continuously on variations in the probability distributions of material parameters and hence the model is stable with respect to the statistical information.

Most publications on stochastic methods adopt more or less justified hypotheses about statistical properties, like stationarity, ergodicity, or the validity of Gaussian distributions for material properties. As discussed in section 3.4.2, such hypotheses are necessary to obtain sufficient statistical information.

Chapter 4

Theory of Stochastic Partial Differential Equations

If parameters in a PDE are stochastic, a *stochastic partial differential equation* (SPDE) results. The focus of the present review is on randomness of the parameters in the operator, i.e. on stochastic partial differential operators. The SPDEs considered here are hence different from stochastic differential equations (SDEs) (Kloeden and Platen, 1995; Øksendal, 1998) or partial differential equations with white noise forcing (Walsh, 1984; Krée and Soize, 1986; Rozanov, 1998; Potthoff et al., 1998).

Theories on the solution of SPDEs with random operators were published by Holden et al. (1996), Benth and Gjerde (1998) and Besold (2000). There, the random parameters were assumed to have infinite variance. Other investigations on SPDEs with random operators but with finite variance random fields were published by Babuška et al. (2002b); Deb et al. (2001); Babuška and Chatzipantelidis (2002); Babuška and Liu (2003); Babuška et al. (2002a). A theory for nonlinear SPDEs was published by Matthies and Keese (2003) and Keese and Matthies (2003d).

A case not discussed in this review is an uncertain geometry. Babuška and Chleboun (2001) investigate this case for a non-stochastic model of uncertainty. They discuss that for general boundary conditions the straightforward idea of constructing a series of problems defined on geometries converging to the correct geometry yields a model converging to a wrong solution. While their model is non-stochastic, their observations need to be taken into account if stochastic geometries are considered.

4.1 Partial Differential Equations with Stochastic Coefficients

In the following, solutions of SPDEs are discussed for a stochastic elliptic boundary value problem as model kept simple for clarity; the theory presented may be extended to more general elliptic SPDEs and to the case of more general (possibly stochastic) boundary conditions in a straightforward manner.

Denote by $\kappa(x,\omega)$ and $f(x,\omega)$ random fields on a domain $R \subset \mathbb{R}^d$ and a suitable probability space $(\Omega, \mathcal{B}, P)$. Then a simple elliptic SPDE is given by

$$\begin{aligned} -\nabla \cdot (\kappa(x,\omega)\nabla u(x,\omega)) &= f(x,\omega), & x \in R, \\ u(x,\omega) &= 0, & x \in \partial R, \end{aligned} \tag{4.1.1}$$

where the solution $u(x,\omega)$ is a random field and where $\omega \in \Omega$. As for usual PDEs, the SPDE is interpreted in a variational sense.

The articles discussed below extend the usual theory of elliptic partial differential equations (e.g. Evans, 1998) to this class of SPDEs by casting the problem in weak form on a suitable Hilbert space and by applying the Lax-Milgram lemma.

Writing the differential operator in Eq. (4.1.1) as a κ-dependent operator T_κ on an appropriate Hilbert space $\mathcal{H}$, the SPDE may be written for $u \in \mathcal{H}$ as the equations

$$\begin{aligned} T_\kappa : \mathcal{H} &\longrightarrow \mathcal{H}^* \\ T_\kappa u &= f. \end{aligned} \tag{4.1.2}$$

Publications investigating the well-posedness of this SPDE differ mainly by the conditions imposed on κ and f. These determine what Hilbert space $\mathcal{H}$ is to be used. The articles discussed here construct $\mathcal{H}$ as a tensor product

$$\mathcal{H} = H_0^1(R) \otimes (S), \tag{4.1.3}$$

where $H_0^1(R)$ is the usual Sobolev space (e.g. Oden and Demkowicz, 1996) describing the spatial part of the solution and where (S) is a space of stochastic (possibly generalised) functions taking the stochastic regularity into account (e.g. Hida et al., 1993).

4.2 Stochastic Partial Differential Equations with Generalised Random Fields

In applications, generalised random fields may be required for modelling the material parameters (e.g. Christakos, 1992), i.e. distributions. First, this case is briefly discussed. Then the case of ordinary random fields is discussed in more detail.

4.2.1 Stochastic Partial Differential Equations with Usual Product

If κ is a generalised random field, then u may also be a generalised random field. Care must then be taken of how to define the product $\kappa(x,\omega)\nabla u(x,\omega)$.

For a Gaussian probability space $(\Omega,\mathcal{B},P)$, Benth and Gjerde (1998) consider the case where (S) is a Hida-Kondratiev distribution space $(S)=(S)^{-\rho,-q,1}$ (see section 2.3). They assume that the weak form associated with the SPDE is coercive and bounded on $H_0^1(R)\otimes(S)^{-\rho,-q,1}$ and apply the Lax-Milgram lemma to show existence and uniqueness of solutions. The case where the solution has finite variance, $(S)=L_2(\Omega)$, is included in this work. Their findings are more general than the other findings discussed below, and their theory includes both the usual product and the Wick-product discussed below. However, the question of how the product between κ and ∇u is defined is not addressed in this work.

This question was addressed by Besold (2000): if $\kappa(x,\omega)$ is a stochastic distribution, then it is not clear how the pointwise multiplication in the expression $\kappa(x,\omega)\nabla u(x,\omega)$ is defined. Even if $u(x,\omega)$ is a stochastic test function, the multiplication operator is in general not continuous and may not be extended to the case where u is a distribution. Besold (2000) shows that the pointwise multiplication may be made well-defined by requiring that $\kappa\in C^\infty(R)\otimes(\mathcal{E})_\infty$, where $(\mathcal{E})_\infty$ is a space of random variables with certain stochastic smoothness requirements (see his thesis for details). Additionally, conditions guaranteeing coercivity and boundedness are imposed. It is shown then by applying the Lax-Milgram lemma that for $\mathcal{H}=H_0^1(R)\otimes(S)^p$ a unique solution of the Eq. (4.1.2) exists, where $(S)^p$ is the Kondratiev distribution space (see section 2.3). The case $(S)=L^2(\Omega)$ is included in this theory.

It is not obvious how the $C^\infty(R)$-regularity on the spatial part of κ may be waived in Besold's theory. One would expect less regularity restrictions on κ's spatial part from usual PDE-theory; e.g. one would expect that the spatial part is allowed to be in $L^\infty(R)$.

4.2.2 Wick Stochastic Partial Differential Equations

The problems in defining the product of distributions are overcome by Holden et al. (1996) by interpreting products between generalised random fields as Wick products, denoted here as $\odot$. Wick SPDEs may be solved by transforming them to deterministic PDEs by the Hermite transform or by the S-transform (see Holden et al. (1996) for details).

SPDEs in the Wick sense have a different interpretation than SPDEs in the usual product sense, because $\mathbf{E}(u\odot v)=\mathbf{E}(u)\mathbf{E}(v)$. As a consequence, for (e.g. linear) Wick SPDEs, the mean of the solution is not influenced by higher statistical

moments of the material parameters.

This behaviour does not agree with the results of homogenisation theory (e.g. Cioranescu and Donato, 1999; Nemat-Nasser and Hori, 1993) and seems not to coincide with the usual interpretation of a stochastic system as a set of possible worlds across which statistics are taken, just as in Monte Carlo simulations. Of course, Monte Carlo simulations of stochastic distributions cannot be done in the usual way, as they do not have realisations as required in a Monte Carlo setting. But even if SPDEs containing ordinary random fields are interpreted in the Wick sense, the results do not coincide with Monte Carlo simulations.

For a discussion comparing the Wick product and the usual product for a stochastic flow problem with the Wick exponential of smoothed white noise as hydraulic conductivity see Holden et al. (1996, pp.128ff.).

Note that the numerical solution of the Wick version of Eq. (4.1.1) by the SFEM techniques of section 6.2 requires considerably less effort than the solution of SPDEs with usual product as the equations resulting for the Wick SPDE are block triagonal (Theting, 2000).

4.3 Stochastic Partial Differential Equations with Ordinary Random Fields

4.3.1 Existence of solutions

Deb et al. (2001); Babuška et al. (2002b) and Babuška and Chatzipantelidis (2002) investigate the special case, where all random fields are ordinary random fields. They choose $(S) := L^2(\Omega)$ and directly extend the usual theory of PDEs to show existence and uniqueness of solutions (using the Lax-Milgram lemma).

Apart from measurability conditions on κ and f, they require that κ is uniformly bounded from above and below, i.e. that there are $\kappa_{min}, \kappa_{max} \in \mathbb{R}$ such that

$$0 < \kappa_{min} < \kappa(x,\omega) < \kappa_{max} < \infty, \qquad \text{a.e. on } R \times \Omega. \tag{4.3.1}$$

The boundedness away from zero is essential. To see this, assume that for every $\varepsilon > 0$ a region $R_0 \subset R$ with positive measure exists on that $P\{|\kappa(x,\omega)| < \varepsilon\} > 0, x \in R_0$. Then the solution of Eq. (4.1.1) has infinite variance. Note that many publications solving elliptic SPDEs use Gaussian random fields and hence do not satisfy this condition.

The variational form of Eq. (4.1.2) on $\mathcal{H}$ is to find $u \in \mathcal{H}$, such that $\mathcal{B}(u,v) = \langle f, v\rangle$ for all $v \in \mathcal{H}$, where the bilinear form

$$\mathcal{B}(u,v) = \int_\Omega \int_R \kappa(x,\omega) \nabla_x u(x,\omega) \cdot \nabla_x v(x,\omega)\, dx\, d\omega, \qquad u,v \in \mathcal{H} \tag{4.3.2}$$

is coercive and bounded due to Eq. (4.3.1). The existence and uniqueness of the weak solution as well as its continuous dependence on the data follow from the Lax-Milgram lemma (e.g. Oden and Demkowicz, 1996) and the second Strang lemma (Ciarlet, 1978; Strang and Fix, 1988).

4.3.2 Perturbations of the Random Fields

For discretisations, an approximation of κ is required. Deb et al. (2001), Babuška et al. (2002b), Babuška and Chatzipantelidis (2002) consider the series expansion Eq. (3.2.4), which we discuss in section 3.2.2 and section 5.1.2,

$$\kappa_m(x,\boldsymbol{\omega}) = \mu_\kappa(x) + \sum_{i=1}^{m} k_i(x)\,\omega_i, \tag{4.3.3}$$

where $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_m)$ is a vector of mutually independent random variables. Hence, the expansion is more specific than the KL-expansion discussed in section 5.1.2. As mentioned in section 3.2.2, prescribing the marginal distribution of κ_m is difficult in this model. However, one could instead use the representations Eq. (5.3.1) and the findings would still be applicable with minor modifications.

If this approximation is used instead of κ, or if an approximation $\hat{f}$ is used instead of f, a perturbed SPDE is solved, yielding a perturbed solution $\hat{u}_m$,

$$T_{\kappa_m}\hat{u}_m = -\nabla\cdot(\kappa_m\nabla\hat{u}_m) = \hat{f}. \tag{4.3.4}$$

To obtain a meaningful numerical solution, it is important that the error $u-\hat{u}_m$ introduced by this perturbation depends continuously on the errors in the approximations of κ and f.

Continuous dependence on perturbed fields If κ_m is uniformly coercive and bounded, then the error is

$$\|u-\hat{u}_m\|_{\mathcal{H}} = O\left(\|f-\hat{f}\|_{\tilde{L}^2(R)} + \|\kappa-\kappa_m\|_{\tilde{L}^\infty(R)}\right), \tag{4.3.5}$$

and hence solutions of the perturbed SPDE converge to the the exact exact solution for suitable approximations of κ and f (Babuška et al., 2002b; Babuška and Chatzipantelidis, 2002). The continuous dependence on f is due to the Lax-Milgram lemma and the continuous dependence on κ is due to the second Strang-lemma (Ciarlet, 1978; Strang and Fix, 1988).

Continuous dependence on probability distributions Babuška and Chatzipantelidis (2002) argue that based on experimental data, more than one probability

distribution is possible for $\kappa(x,\boldsymbol{\omega})$. Consequently, there is an uncertainty in the probability distributions of the independent random variables ω_i and a perturbation of the field

$$\hat{\kappa}_m = \mu_\kappa(x) + \sum_{n=1}^{m} \kappa_i(x)\hat{\omega}_i \tag{4.3.6}$$

needs to be considered, where it is assumed that $\text{range}(\omega_i) = \text{range}(\hat{\omega}_i) =: \Omega_i$ and that the probability densities $p_{\boldsymbol{\omega}}$ and $p_{\hat{\boldsymbol{\omega}}}$ of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ and $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \ldots, \hat{\omega}_m)$ exist and are everywhere positive. Let u_m be the solution of $T_{\kappa_m} u_m = f$ and $\hat{u}_m$ be the solution of $T_{\hat{\kappa}_m}\hat{u}_m = f$. Let $\Omega^{(m)} = \Omega_1 \times \cdots \times \Omega_m$, then the error is

$$\|u_m - \hat{u}_m\|_{\mathcal{H}} \le C \left\| \frac{p_{\boldsymbol{\omega}} - p_{\hat{\boldsymbol{\omega}}}}{p_{\boldsymbol{\omega}}} \right\|_{L^\infty(\Omega^{(m)})} \left(\|f\|_{\tilde{L}^2(R)} + \|u\|_{\mathcal{H}} \right), \tag{4.3.7}$$

This shows that the solution is enhanced when the statistics are refined from experimental data, and that convergence to the true experimental solution may be achieved, at least theoretically; see Babuška et al. (2002a).

Unsolvable SPDEs due to Perturbation As for the original SPDE, the perturbed SPDE is solvable if κ_m is uniformly coercive and bounded. An unsolvable SPDE may result if κ_m violates this condition. This is demonstrated by Babuška and Chatzipantelidis (2002), who define the function

$$\sigma_0(x;m) := \sum_{i=0}^{m} |\kappa_i(x)| b_i, \tag{4.3.8}$$

where b_i is the diameter of ω_i's range which is assumed to be bounded, $\text{range}\,\omega_i = [-b_i, b_i]$. If

$$\sigma_0(x;m) > \mu_\kappa(x) \tag{4.3.9}$$

on a region $R_0 \subset R$ with positive volume $|R_0| > 0$, then κ_m takes values arbitrarily close to zero with positive probability, and then Eq. (4.3.4) is unsolvable with positive probability (Babuška and Chatzipantelidis, 2002, Theorem 3.1).

For fields with correlation structure $\exp(-c\|x-y\|)$, Babuška and Chatzipantelidis (2002) show that $\sigma_0(x;m)$ grows without bounds for $m \longrightarrow \infty$, even though $\kappa_m(x,\omega)$ converges uniformly to $\kappa(x,\omega)$ in variance. This covariance is used quite frequently in publications, but the number of terms considered is usually so small that the perturbed problem is solvable.

However, in conjunction with a Galerkin projection in the stochastic dimensions, this effect does not occur (Matthies and Keese, 2003). The reason is that the Galerkin method may be performed without explicitly representing the random fields in a finite number of random variables—in a Galerkin method, this is performed implicitly by choosing an ansatz in a finite number of random variables.

4.4 Conclusions

The theories of SPDEs presented above differ in the kind of random fields allowed. When material parameters are modelled by generalised random fields, then the theory is complex as then the problem of how to define the products of generalised random fields has to be overcome.

Holden et al. (1996) consider random fields as stochastic distributions and interpret products between random fields as Wick-products. They provide a theory of SPDEs for this case. The main shortcoming of their theory is that—e.g. for linear problems—higher statistical moments of system parameters do not influence the mean of the answer, a contradiction to the results of homogenisation theory. Also, they require the existence of strong solutions, which results in considerable restrictions on the boundary conditions and source terms. These may be relaxed by a variational formulation (Matthies and Bucher, 1999; Theting, 2000), but nonetheless the Wick product seems not to be the right model for the problems aimed at here.

If the products between (generalised) random fields are instead interpreted in the usual sense, stronger stochastic regularity is required. This has been investigated by (Besold, 2000) who provides a variational theory of SPDE as operators on tensor product spaces of stochastic distributions with the usual Sobolev spaces. While this work allows material parameters to be distributions in the stochastic dimension, a shortcoming is that they are required to be smooth in the spatial dimension. The stochastic regularity of the solution determines the convergence rate of numerical approximations, and a variational theory for this was earlier devised by Benth and Gjerde (1998).

The theories by Babuška et al. (2002b); Deb et al. (2001); Babuška and Chatzipantelidis (2002); Babuška and Liu (2003); Babuška et al. (2002a) are a direct extension of usual elliptic theory and as such they require random fields to have finite variance, but they allow the same spatial irregularity as in usual (deterministic) elliptic theory.

As conclusions, the theories of SPDEs outlined in the previous sections are not yet completely satisfactory—of the presented works, Babuška et al. (2002b) is the closest to the usual theory of (deterministic) PDEs and has the usual restrictions on the spatial regularity of the random fields, but it requires the random fields to depend on only a finite number of independent random variables, and only ordinary random fields with finite variance are allowed. Benth and Gjerde (1998) do not investigate conditions that have to hold for the parameters, but directly assume coerciveness and boundedness. The theory by Holden et al. (1996) allows a high stochastic irregularity, but interprets products between random fields as Wick products and thus seems not to be the right model for the problems aimed at here. The theory by Besold (2000) allows to use (more regular) stochastic

distributions and interprets products between random fields in the usual sense; but here a high regularity in the spatial dimension is required for the random fields in the operator.

Part II

Discretisation of Stochastic Partial Differential Equations (Stochastic Finite Elements)

Chapter 5

Discretisation of Random fields

In a numerical setting, random fields need to be discretised both in the stochastic and in the spatial dimension. The stochastic discretisation yields a representation in a finite number of independent random variables and is usually performed by first representing the random field in a countable number of random variables and then keeping only a finite number of these. The stochastic discretisation often involves a spatial discretisation, e.g. by standard techniques, like finite elements. Most discretisation techniques may be regarded as special cases of some more abstract techniques and are presented here as such.

A classic on random fields including simple discretisation techniques is Vanmarcke (1988). Details on discretisation techniques may be found in Grigoriu (1995), and with an emphasis on spectral models for Monte Carlo methods in Ogorodnikov and Prigarin (1996), or in Prigarin (2001).

5.1 Series Representations

In the following, let κ be a random field on a compact region $R \subset \mathbb{R}^d$ and on a probability space $(\Omega, \mathcal{B}, P)$; see section 3.1 for details.

The stochastic field may be expanded as a series

$$\kappa_m(x,\omega) = \sum_{i=1}^{m} N_i(x)\kappa_i(\omega) = \boldsymbol{N}(x)\boldsymbol{\kappa}(\omega) \tag{5.1.1}$$

of a finite number of random variables κ_i and functions $N_i : R \to \mathbb{R}$ which are collected in the vectors $\boldsymbol{\kappa}(\omega) = (\kappa_1(\omega), \ldots, \kappa_m(\omega))^T$ and $\boldsymbol{N}(x) = (N_1(x), \ldots, N_m(x))$. Often, the functions N_i are finite element shape functions.

The mean and covariance of the approximated field are

$$\mu_{\kappa_m}(x) = \boldsymbol{N}(x)^T \boldsymbol{\mu}_{\boldsymbol{\kappa}} \tag{5.1.2}$$

$$\operatorname{cov}_{\kappa_m}(x,y) = \boldsymbol{N}(x)^T \boldsymbol{C}_{\boldsymbol{\kappa}} \boldsymbol{N}(y), \tag{5.1.3}$$

where $\boldsymbol{\mu}_{\boldsymbol{\kappa}} = \mathbf{E}(\boldsymbol{\kappa}(\omega))$ is the mean vector and where $\boldsymbol{C}_{\boldsymbol{\kappa}}$ is the covariance matrix of the random vector $\boldsymbol{\kappa}(\omega)$.

5.1.1 Overview of Series Representations

Most methods presented below were already discussed by Matthies et al. (1997); they are presented here as special cases of Eq. (5.1.1):

The interpolation method or shape function method (Liu et al., 1986a,b) interpolates the random field in finite element shape functions $N_i(x)$ in nodal positions x_j. Assume that a nodal basis is used, i.e. $N_i(x_j) = \delta_{ij}$. Then $\kappa_i = \kappa(x_i)$, and the mean and covariance of the approximation may be computed from Eqs. (5.1.2–5.1.3) with

$$(\boldsymbol{\mu}_{\boldsymbol{\kappa}})_i = \mu_\kappa(x_i) \quad \text{and} \quad (\boldsymbol{C}_{\boldsymbol{\kappa}})_{i,j} = \operatorname{cov}_\kappa(x_i, x_j). \tag{5.1.4}$$

The midpoint method (Der Kiureghian and Ke) is a special case of the shape function method, where $N_i(x)$ is piecewise constant and where x_i is the midpoint of the elements. The midpoint method is sometimes claimed to overestimate the field's variability. However, this claim seems not to be justified as the approximate covariance is simply an interpolation of the exact covariance as Eqs. (5.1.2–5.1.3) and Eq. (5.1.4) show.

The spatial average method (Vanmarcke and Grigoriu, 1983) also uses piecewise constant functions $N_i(x)$, but chooses κ_i as the spatial average of $\kappa(x,\omega)$ over the i-th domain in the finite element discretisation: $\kappa_i(\omega) = \frac{1}{|R_i|}\int_{R_i} \kappa(x,\omega)\,dx$, where $R_i = \operatorname{supp} N_i$ with volume $|R_i|$. The mean and covariance of the approximation may be computed from Eq. (5.1.2) with

$$(\boldsymbol{\mu}_{\boldsymbol{\kappa}})_i = \frac{1}{|R_i|}\int_{R_i} \mu_\kappa(x)\,dx \quad \text{and} \quad (\boldsymbol{C}_{\boldsymbol{\kappa}})_{ij} = \frac{1}{|R_i|\,|R_j|}\int_{R_i}\int_{R_j} \operatorname{cov}_\kappa(x,y)\,dx\,dy. \tag{5.1.5}$$

The mean and covariance of the approximation are weighted averages of the exact statistics and hence this method underestimates the variability of κ. Li and Kiureghian (1993) find the spatial averaging method to be superior to the midpoint method for several types of covariance functions.

If κ is a Gaussian field, then κ_i is Gaussian also. A disadvantage of the spatial average method is that the probability distribution of κ_i is difficult to obtain for non-Gaussian κ due to the spatial weighting.

The orthogonal series expansion chooses functions $N_i(x)$ in Eq. (5.1.1), which are mutually orthogonal in $L^2(R)$. Then

$$\kappa_i(\omega) = \frac{(\kappa, N_i)_{L^2(R)}}{\|N_i\|_{L^2(R)}} = \frac{1}{\|N_i\|_{L^2(R)}} \int_R \kappa(x,\omega) N_i(x)\, dx, \tag{5.1.6}$$

and the mean and the covariance may be computed from Eq. (5.1.2) with

$$\boldsymbol{\mu}_{\boldsymbol{\kappa}} = \int_R \mu_\kappa(x) \boldsymbol{N}(x)\, dx \quad \text{and} \quad \boldsymbol{C}_{\boldsymbol{\kappa}} = \int_R \int_R \boldsymbol{N}(x)\, \mathrm{cov}_\kappa(x,y)\, \boldsymbol{N}(y)^T\, dx\, dy. \tag{5.1.7}$$

As the approximated statistics are weighted averages of the exact quantities it is difficult to obtain the probability distribution of κ_i for a non-Gaussian κ.

Orthogonal functions are known only for special geometries. For general geometries, they may be obtained numerically, but then the Karhunen–Loève expansion is superior.

The Karhunen–Loève expansion (KL-expansion) will be discussed in section 5.1.2. It is is a special case of the orthogonal series expansion, where the N_i are chosen as eigenfunctions of a Fredholm equation with the covariance as kernel.

Spectral Representations expand κ as a sum of trigonometric functions with random phase angles and random amplitudes. For overviews see e.g. the text books Ogorodnikov and Prigarin (1996); Prigarin (2001), or the brief overview in Shinozuka and Deodatis (1997). Spectral representations are primarily used to generate samples according to given spectral power densities in Monte Carlo simulations. Note that the KL-expansion of a field defined on the whole space $\mathbb{R}^d$ is a spectral representation.

5.1.2 The Karhunen–Loève Expansion

The Karhunen–Loève expansion (KL-expansion) is presented in its own section as its use is very common. It was invented independently by Karhunen (1947), Loève (1948), and Kac and Siegert (1947) and is discussed in many text books, e.g. in Van Trees (1968); Adler (1981); Vanmarcke (1988); Christakos (1992); Papoulis (1991); Ghanem and Spanos (1991b).

The KL-eigenvalue problem: Let $\kappa : R \times \Omega \longrightarrow \mathbb{R}$ be a random field with continuous and bounded covariance function $\mathrm{cov}_\kappa(x,y)$. Define the Fredholm operator T with the covariance as kernel

$$\begin{array}{rcl} T : L^2(R) & \longrightarrow & L^2(R) \\ (Tu)(x) & := & \int_R \mathrm{cov}_\kappa(x,y)u(y)\,dy. \end{array} \tag{5.1.8}$$

As cov_κ is bounded and symmetric, T is compact and selfadjoint. As covariance functions are positive semi-definite, so is T. Hence, solutions of the eigenvalue problem

$$Tk_i = \lambda_i k_i, \qquad k_i \in L^2(R), i \in \mathbb{N}, \tag{5.1.9}$$

(a Fredholm integral equation of the second kind), have the following properties, e.g. Werner (1997, pp. 229ff.) or Adler (1981):

- The eigenvalues λ_i are real and can be ordered as $\|T\| = \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. They have the property $\|C\|^2_{L^2(R\times R)} = \int_R \int_R |C(x,y)|^2\,dx\,dy = \sum_i \lambda_i^2 = \|T\|^2_{HS}$.
- The eigenfunctions $k_i(x)$ are continuous and mutually $L^2(R)$-orthogonal.
- As cov_κ is continuous, it follows from Mercer's Theorem (e.g. Werner, 1997, Theorem VI.4.2.) that $\mathrm{cov}_{\kappa_m}(x,y) := \sum_{i=1}^m \lambda_i \kappa_i(x)\kappa_i(y)$ converges to the exact covariance function $\mathrm{cov}_\kappa(x,y)$ absolutely and uniformly on $R \times R$, hence $\|\,\mathrm{cov}_\kappa - \mathrm{cov}_{\kappa_m}\,\|_{L^2(R\times R)} \to 0$ as $m \longrightarrow \infty$.

The Karhunen–Loève expansion is the series

$$\kappa(x,\omega) = \mathbf{E}\left(\kappa(x)\right) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\xi_i(\omega)\,\kappa_i(x), \qquad \text{where} \tag{5.1.10}$$

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}}\left(\kappa - \mathbf{E}\left(\kappa\right), k_i\right)_{L_2(R)} = \frac{1}{\sqrt{\lambda_i}} \int_R \Big(\kappa(x,\omega) - \mathbf{E}\left(\kappa(x)\right)\Big) k_i(x)\,dx. \tag{5.1.11}$$

The random variables κ_i are mutually uncorrelated and centred with unit variance, and the truncated Karhunen–Loève series

$$\kappa_m(x,\omega) := \mathbf{E}\left(\kappa(x)\right) + \sum_{i=1}^{m} \sqrt{\lambda_i}\,\xi_i(\omega)\,\kappa_i(x) \tag{5.1.12}$$

converges to κ in variance uniformly in x (by Mercer's theorem),

$$\sup_{x\in R} \mathbf{E}\left((\kappa(x)-\kappa_m(x))^2\right) = \sup_{x\in R} \sum_{i=m+1}^{\infty} \lambda_i \kappa_i(x)^2 \longrightarrow 0, \qquad \text{as } m \longrightarrow \infty. \tag{5.1.13}$$

The KL-expansion is an optimal representation of κ. For any other linear combination of m functions, the error $\|\kappa-\kappa_m\|_{L^2(\Omega\times R)}$ is not smaller than for the KL-expansion (Ghanem and Spanos, 1991b). However, nonlinear approximations may yield better approximations (Cohen and D'Ales, 1997).

If κ is Gaussian, then the κ_i are Gaussian random variables and hence mutually independent. For non-Gaussian fields, the distributions of the κ_i are not known analytically and may be computed numerically from Eq. (5.1.11). If κ is non-Gaussian, then the κ_i are only uncorrelated but not independent, which makes working with them more difficult and will be discussed in section 5.2.

Discrete Karhunen–Loève Expansion: Exact solutions for the Fredholm integral equation of the second kind Eq. (5.1.9) are only known for special geometries and for special covariance functions. For problems in one dimension and for some types of covariance functions, exact solutions were given e.g. by Van Trees (1968) and by Ghanem and Spanos (1991b).

In general, numerical approximations are required, e.g. by the methods presented in Atkinson (1997); Hackbusch (1995); Press et al. (1997). Usually, only the most prominent eigenvectors are required and Lanzcos type methods may be used for this. Efficient implementations of such methods are readily available (e.g. Lehoucq et al., 1998; Maschhoff and Sorensen, 1996).

A convergence study of the truncated Karhunen–Loève expansion for stochastic processes ($R \subset \mathbb{R}^1$) was performed by Huang et al. (2001) by varying the size of R, the correlation length, the type of covariance function, and the number of KL-terms. The findings were:

- The smoother the covariance function, the less KL-terms are needed. This is plausible as the KL-expansion is a kind of generalised Fourier transform.
- The larger the ratio of correlation length to the size of the domain, the fewer KL-terms are needed. This may be visualised by noting that the faster $\mathrm{cov}_\kappa(x,y)$ decays for increasing $\|x-y\|$, the smaller is the correlation length. A small value of $C(x,y)$ means that the field at x is almost uncorrelated from its value at y. It is plausible that a good representation of the field requires a number of uncorrelated random variables large enough to cover the whole region R by patches whose size grow with the correlation length (e.g. with balls whose radius is half the correlation length).

- It is claimed that the analytical solution of Eq. (5.1.9) gives significantly better approximations of κ than when approximate solutions to the eigenvalue problem are used.

  Based upon this statement, Huang et al. (2001) conclude that the non-availability of analytic solutions limits the applicability of the Karhunen–Loève expansion as the approximate solutions require more terms in the KL-series than the analytical solution and hence more stochastic dimensions (see section 5.2). A similar view is expressed by Li and Kiureghian (1993) who state that the KL-method reduces to the shape function method with a KL-expansion of the discrete covariance if the eigenvalue problem is solved approximately.

  Both claims hold true only if the numerical solution of the eigenvalue problem is performed on the same (or coarser) ansatz space as used for the representation of the field. But this is not necessarily so—the numerical solution converges to the analytical solution for finer and finer ansatz spaces (e.g. Atkinson, 1997; Hackbusch, 1995). The eigenvectors obtained on a sufficiently fine mesh may be interpolated to a coarser mesh for the representation of the stochastic field. If one proceeds like this, no drawbacks in using approximate solutions to the KL-problem are to be expected.

The variance of the truncated KL-expansion is smaller than the variance of κ. This can be remedied by multiplying the expansion by normalising constant (Ghanem, 1998c), but this leads to a larger variance of the error in the approximation.

It is sometimes suggested to omit the approximate solution of the KL eigenvalue problem and restrict solutions of a problem on a larger region, for which exact solutions are known—e.g. a square—to the region of interest. However, this leads to a non-optimal representation as the scalar products and hence the eigenfunctions and eigenvalues are different for L^2-functions on different spatial domains. Hence, more terms in the expansion would be required and the resulting larger stochastic dimension might then increase the further costs of computations considerably.

In the experience of the author, the costs in numerically computing the KL-expansion are not the limiting factor in obtaining stochastic finite element solutions. In the opinion of the author, the advantages of using the KL-expansion (optimal expansion and hence minimisation of the number of stochastic dimensions) outweigh their disadvantages (need to solve an eigenvalue problem in $L^2(R)$).

5.2 Representation in Independent Random Variables

All techniques discussed in the two previous sections represent $\kappa(x,\omega)$ in a finite number of possibly correlated random variables $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_m)$ with mean vector $\boldsymbol{\mu}_\kappa$ and covariance matrix $\boldsymbol{C}_\kappa$, where the κ_i are Gaussian if and only if the random field is Gaussian.

For the further numerical treatment, a representation in a finite number of *independent* random variables is required. These may be seen as coordinate axes, and the representation of random properties in m mutually independent random variables may be identified with a representation in a probability space, where the space of elementary events is a subset of $\mathbb{R}^m$ equipped with the probability measure induced by the m random variables; see section 5.3.

5.2.1 The Gaussian Case:

A representation in independent random variables may be obtained for a Gaussian field by linear transformations of $\boldsymbol{\kappa}$, because Gaussian random variables are independent if and only if they are uncorrelated, i.e. iff $\boldsymbol{C}_\kappa$ is a diagonal matrix.

If the Karhunen–Loève decomposition has been used for the discretisation, then the κ_i are already uncorrelated. Otherwise, the random variables need to be transformed so that the covariance matrix becomes diagonal. In principle, any diagonalisation technique from linear algebra may be used for this.

A Cholesky decomposition may be used to obtain a basis of independent random variables: A symmetric positive definite $\boldsymbol{C}_\kappa$ has a Cholesky decomposition $\boldsymbol{C}_\kappa = \boldsymbol{A}\boldsymbol{A}^*$, where $\boldsymbol{\omega} := \boldsymbol{A}^{-1}\boldsymbol{\kappa}$ is a vector of uncorrelated random variables. Hence, if κ is Gaussian, then $\boldsymbol{\kappa} = \boldsymbol{A}\boldsymbol{\omega}$, where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^T$ is a vector of independent standard Gaussian random variables $\omega_i = \mathcal{N}(0,1), i = 1, \ldots, m$.

Often $\boldsymbol{C}_\kappa$ has a special structure. For example, if κ is stationary and is sampled on a regular lattice, then $\boldsymbol{C}_\kappa$ is block-Toeplitz. For such special cases, efficient algorithms for the Cholesky decomposition applicable to large numbers of random variables were presented by Ogorodnikov and Prigarin (1996).

5.2.2 The Non-Gaussian Case

Generally speaking, if $\boldsymbol{\kappa}$ contains non-Gaussian random variables, then it is difficult to obtain a representation in independent random variables, as then uncorrelatedness and independence are not equivalent.

Obtaining a representation in independent random variables for a non-Gaussian field $\kappa(x,\omega)$ usually involves nonlinear transformations of random variables; a diagonalisation of $\boldsymbol{C}_\kappa$ is not sufficient. To obtain a representation in independent RVs, one may transform $\boldsymbol{\kappa}$ to Gaussian RVs and orthogonalise these. A standard

technique for this (e.g. Melchers, 1999) is the Rosenblatt transformation (Rosenblatt, 1952), which represents the joint distribution as a product of conditional probability distributions and then transforms each of these to a Gaussian distribution.

In the following, only procedures representing Gaussian fields in independent Gaussian random variables will be considered. If non-Gaussian random fields are expressed as nonlinear transformations of Gaussian ones (see section 3.2.2), these methods may be sufficient.

As we already saw in section 2.2.3, non-Gaussian random fields may also be expanded in polynomial chaos. A polynomial chaos expansion is not used to obtain a representation in independent random variables but is used after a set of independent variables has been obtained; we will discuss it in more detail in section 6.2.1.

5.2.3 Dimension Reduction and Optimal Representations

As mentioned above, once the problem is described in m independent random variables, it may be represented in a probability space where the space of elementary events is a subset of $\mathbb{R}^m$, and where the probability measure is the measure induced by the independent random variables; see section 5.3. Every random variable contributes one dimension and it is desirable to keep their number small.

The discretisation techniques discussed in section 5.1.1 may result in a representation in many random variables, e.g. the interpolation method results in one random variable for each node in the finite element mesh. The KL-expansion offers a natural way to reduce the number of random variables.

Principal Component Analysis, Discrete KL-expansion If another discretisation technique than the KL-expansion has been used, then the discrete equivalent to the KL decomposition may be used to reduce the dimensions: The random vector $\boldsymbol{\kappa}(\omega)$ may be expanded in the eigenvectors of the covariance matrix $\boldsymbol{C}_{\boldsymbol{\kappa}}$. By keeping only the components belonging to the largest eigenvalues, a representation in a smaller number of uncorrelated random variables is obtained.

Optimisation of Representation Once $\kappa(x,\omega)$ has been approximated in a finite number of random variables, techniques for enhancing the approximation may be applied, e.g. the optimal linear estimation method (OLE) (Li and Kiureghian, 1993), which projects the random field on a set of random variables; this method is also called Kriging (Matheron, 1963; Sobczyk and Kirner, 2001)

These methods represents a random field in a set of random variables $\boldsymbol{\kappa}$ like in Eq. (5.1.1) as $\kappa_m(x,\omega) = \sum_{i=1}^{m} N_i(x)\kappa_i(\omega)$, but do not choose the spatial functions

$N_i(x)$ a priori. Instead, they are chosen so that κ_m is an optimal unbiased estimator of κ, resulting in

$$\kappa(x,\omega) = \mathbf{E}\left(\kappa(x)\right) + \boldsymbol{C}^T_{\kappa(x)\boldsymbol{\kappa}}\boldsymbol{C}^{-1}_{\boldsymbol{\kappa}}(\boldsymbol{\kappa}-\boldsymbol{\mu}), \tag{5.2.1}$$

where $\boldsymbol{\mu}$ is the mean of the random field at the finite element nodes, $\boldsymbol{C}_{\kappa(x)\boldsymbol{\kappa}}$ is the cross-covariance between the random field and the random variables, and $\boldsymbol{C}_{\boldsymbol{\kappa}}$ is the covariance matrix of the random vector $\boldsymbol{\kappa}$. Li and Kiureghian (1993) find that the Gaussian random fields with covariance functions Eq. (3.4.9) are approximated with negligible error if the element size is half the correlation length (or smaller). They note the error of the KL-approximation is large at the borders of the region, while it is distributed more uniformly over the region if the OLE method is used.

This technique was developed as a discretisation method, but it may be combined with any of the other methods discussed above. Each of them yields a vector of random variables. Once this vector is available, the OLE may be used to obtain an optimal representation. For example, the EOLE (Expansion OLE) method (Li and Kiureghian, 1993) combines the OLE method and the KL-expansion.

5.3 Finite Dimensional Elementary Event Space

Once a random field has been represented in a finite number of independent random variables $\omega_1,\ldots,\omega_m$, it may be seen as a function on a high dimensional space as was discussed by Deb et al. (2001); Babuška and Chatzipantelidis (2002); Babuška et al. (2002b). The independence of the underlying random variables allows to see each of them as the axis of a coordinate system. This is asserted by the Doob-Dynkin lemma (e.g. Øksendal, 1998) which states that the space of all $\sigma(\omega_1,\ldots,\omega_m)$-measurable random variables may be identified with the set of measurable functions $g(\omega_1,\ldots,\omega_m)$.

Consider the case of Eq. (3.2.1), where the random field $\kappa(x,\omega) = \phi(x,\gamma(x,\omega))$ is a nonlinear transformation of a Gaussian random field γ. Approximate γ (e.g. by any technique described above) as $\gamma_m(x,\boldsymbol{\omega}) = \sum_{i=1}^m \sqrt{\lambda_i}k_i(x)\omega_i$ with mutually independent standard Gaussian random variables $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_m)$. Then κ may be approximated as

$$\kappa_m(x,\boldsymbol{\omega}) = \phi\Big(x, \sum_{i=1}^m \sqrt{\lambda_i}k_i(x)\omega_i\Big), \quad x \in R. \tag{5.3.1}$$

One may thus identify κ_m with a random field on the probability space

$$(\Omega^{(m)}, \mathcal{B}^{(m)}, P_m), \qquad \text{with } \Omega^{(m)} = \mathbb{R}^m, \tag{5.3.2}$$

where $\mathcal{B}^{(m)}$ is the m-dimensional Borel σ-algebra, and where

$$dP_m(\boldsymbol{\omega}) = (2\pi)^{-m} \exp(-|\boldsymbol{\omega}|^2/2) d\boldsymbol{\omega}, \qquad \boldsymbol{\omega} \in \mathbb{R}^m \tag{5.3.3}$$

is the m-dimensional Gaussian probability measure ($d\boldsymbol{\omega}$ denotes the Lebesgue measure).

The construction can be generalised in a straightforward manner to the case where the independent random variables ω_i are non-Gaussian. Such representations were used by Deb et al. (2001); Babuška and Chatzipantelidis (2002) and by Babuška et al. (2002b). For general real-valued independent random variables $\omega_1, \ldots, \omega_m$, the probability space may be identified with $(\Omega^{(m)}, \mathcal{B}^{(m)}, P_m)$, where $\Omega^{(m)} = \Omega_1 \times \cdots \times \Omega_m$, where $\Omega_i = \text{range}(\omega_i)$, where $\mathcal{B}^{(m)}$ is the Borel σ-algebra on $\Omega^{(m)}$, and where P_m is the probability distribution of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$, see section 2.1.

5.4 Conclusions

The discretisation of a random field yields a representation as a function of a finite number of mutually independent random variables. This may be identified with a representation on a probability space, where the set of elementary events is a domain in a finite dimensional space, and where the probability measure is the a measure induced by the independent random variables (their distribution).

An approximation in a finite number of independent random variables may be obtained for Gaussian random fields directly by the techniques discussed above. Representations of non-Gaussian random fields in independent random variables are often based on representations in Gaussian random variables.

In the solution of PDEs with stochastic coefficients, the mesh for the discretisation of the stochastic fields may be chosen independently of the mesh for the spatial discretisation. As a rule of thumb, Kuireghian and Ke (1988) propose to choose the mesh size for the random field discretisation approximately as a quarter to a half of the correlation length. Sometimes it is suggested to choose the stochastic mesh size based on the scale of fluctuation (e.g. Haldar and Mahadevan, 2000, p. 186).

The author believes that when a KL-expansion is used, these rules of thumb need to be regarded only implicitly: the mesh for the numerical solution of the discrete KL-problem should be chosen as fine as computational resources permit to obtain the best possible approximation of the KL-eigenvalue problem and thus to obtain a small number of independent random variables $\omega_1, \ldots, \omega_m$. Afterwards, the eigenvectors kept for the KL-expansion may be interpolated to a coarser mesh used for the representation of the random field. The resolution of this coarser mesh may then be chosen based on the smoothness of the KL-eigenvectors kept

in the representation. However, numerical experiments supporting this view have yet to be performed.

Chapter 6

Discretisation of Stochastich Partial Differential Equations

The discretisation of a stochastic partial differential equation (SPDE) is usually performed by the following steps:

1. Discretise the differential operators in space by usual techniques; e.g. by finite elements, see Strang and Fix (1988), Zienkiewicz and Taylor (2000). This yields a semi-discretisation in the spatial dimensions.

2. Represent the semi-discretised SPDE in a finite number of independent basic random variables, e.g. by the techniques of chapter 5.

3. Perform the stochastic discretisation. We concentrate here on techniques that expand the stochastic part of the solution in functions of these basic random variables. Alternatively, statistics may be obtained by numerical integration, for example by Monte Carlo simulations. Techniques for this are discussed in section 7.1.

4. Solve the resulting equations, post-process the obtained solution.

6.1 Semi-Discretisation

The first steps in the above outline yield a semi-discretisation of the SPDE. We discuss this exemplary for the elliptic SPDE Eq. (4.1.1). By discretising the random fields as shown in chapter 5, an approximation in a finite number of m independent random variables $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ is obtained. The SPDE is thus approximated

in a probability space $(\Omega^{(m)}, \mathcal{B}, P_m)$, where $\Omega^{(m)}$ is the finite-dimensional tensor product of the ranges of the ω_i and where P_m is the probability distribution of $\boldsymbol{\omega}$; see section 5.3. In toto, this yields an approximation of the SPDE in the $(d+m)$-dimensional space $R \times \Omega^{(m)}$ (Babuška et al., 2002b; Deb et al., 2001; Babuška and Chatzipantelidis, 2002).

The SPDE Eq. (4.1.1) is then approximated by

$$\begin{aligned} -\nabla_x \cdot \big(\kappa(x,\boldsymbol{\omega})\nabla_x u(x,\boldsymbol{\omega})\big) &= f(x,\boldsymbol{\omega}), & x \in R,\ \boldsymbol{\omega} \in \Omega^{(m)} \\ u(x,\boldsymbol{\omega}) &= 0, & x \in \partial R,\ \boldsymbol{\omega} \in \Omega^{(m)}, \end{aligned} \tag{6.1.1}$$

which is to be understood in the variational sense discussed in chapter 4. Note that here κ and f denote the discretised random fields. In the following, it will be clear from the context and from the letter ω or the bold letter $\boldsymbol{\omega}$, whether the non discretised random fields or their discretisations are meant.

The spatial discretisation may be performed by almost any technique, e.g. by finite differences or by finite elements. It is shown here exemplary for a finite element discretisation (e.g. Strang and Fix, 1988; Zienkiewicz and Taylor, 2000). Denote by $\boldsymbol{N}(x) = (N_1(x), \ldots, N_n(x))^T$ a vector of ansatz functions with $N_i \in H_0^1(R)$. An ansatz for the solution in $\boldsymbol{N}(x)$ yields a semi-discretisation of Eq. (6.1.1). Similarly to the method of lines for instationary boundary value problems, where the coefficients would be time dependent, an expansion

$$u^{semi}(x,\boldsymbol{\omega}) = \sum_{i=1}^{n} u_i(\boldsymbol{\omega}) N_i(x) = \boldsymbol{N}(x)^T \boldsymbol{u}(\boldsymbol{\omega}), \tag{6.1.2}$$

results, where the random variables $\boldsymbol{u}(\boldsymbol{\omega}) = (u_1(\boldsymbol{\omega}), \ldots, u_n(\boldsymbol{\omega}))$ are the to be computed degrees of freedom, and where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m) \in \Omega^{(m)}$. The further stochastic discretisation involves a discretisation of the random variables u_i and will be the topic of the next sections.

Application of Galerkin conditions in the spatial dimension yields the weak form (we ignore here the Dirichlet boundary conditions)

$$\int_R \kappa(x,\boldsymbol{\omega}) \nabla \boldsymbol{N}(x) \nabla \boldsymbol{N}(x)^T \, dx \, \boldsymbol{u}(\boldsymbol{\omega}) = \int_R f(x,\boldsymbol{\omega}) \boldsymbol{N}(x) \, dx, \qquad \boldsymbol{\omega} \in \Omega^{(m)}, \tag{6.1.3}$$

which is a set of linear equations with stochastic coefficients and may be written after eliminating the Dirichlet conditions as

$$\boldsymbol{K}(\boldsymbol{\omega}) \boldsymbol{u}(\boldsymbol{\omega}) = \boldsymbol{f}(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \Omega^{(m)}. \tag{6.1.4}$$

Here, $\boldsymbol{K}(\boldsymbol{\omega}) \in \mathbb{R}^{n \times n}$ is the usual stiffness matrix of the finite element ansatz obtained for a material parameterised by $\boldsymbol{\omega} \in \Omega^{(m)}$. The semi-discretisation of any linear stationary SPDE (including cases with stochastic boundary conditions) can be written in this form.

Similarly, nonlinear stationary SPDEs may be semi-discretised as a set of n nonlinear equations parameterised by $\boldsymbol{\omega}$ (Keese and Matthies, 2003d,b, 2002) as

$$\boldsymbol{r}(\boldsymbol{u}(\boldsymbol{\omega}),\boldsymbol{\omega}) = 0, \qquad \boldsymbol{\omega} \in \Omega^{(m)} \tag{6.1.5}$$

with a nonlinear function $\boldsymbol{r} : \mathbb{R}^K \times \Omega^{(m)} \longrightarrow \mathbb{R}^n \times \Omega^{(m)}$ parameterised by a stochastic parameter $\boldsymbol{\omega}$.

As in the method of lines for deterministic PDEs, the semi-discretisation of instationary SPDEs may be performed using time dependent random coefficients

$$u(t,x,\boldsymbol{\omega}) = \sum_{i=1}^{n} N_i(x) u_i(t,\boldsymbol{\omega}) = \boldsymbol{N}(x)^t \boldsymbol{u}(t,\boldsymbol{\omega}), \tag{6.1.6}$$

and after performing the discretisation, a (possibly nonlinear) system of ODEs parameterised by $\boldsymbol{\omega}$ is obtained,

$$\boldsymbol{r}\big(\dot{\boldsymbol{u}}(t,\boldsymbol{\omega}),\boldsymbol{u}(t,\boldsymbol{\omega}),t,\boldsymbol{\omega}\big) = 0, \qquad \boldsymbol{\omega} \in \Omega^{(m)}. \tag{6.1.7}$$

Existence and uniqueness of solutions as well as a

For instationary SPDEs, the discretisation in time may be performed by usual methods, e.g. by Runge-Kutta schemes. Galerkin discretisation by tensor products in temporal and stochastic ansatz functions for ordinary differential equations with stochastic coefficients were investigated recently (Babuška and Liu, 2003).

6.2 Series Expansions for SPDEs

Various frequently used techniques for the discretisation of SPDEs may be seen as series expansion techniques and are discussed here as such. These methods include certain response surface methods, the Neumann expansion method, perturbation methods, Galerkin methods, and non-intrusive methods based on orthogonal projections.

These techniques expand the solution in the stochastic dimensions in a set $\{H_\beta\}_{\beta\in\mathcal{I}}$ of linearly independent ansatz functions indexed by a finite set $\mathcal{I}$. The ansatz functions H_β are elements of $L^2(\Omega^{(m)})$ (or, more generally, of the space (S) used in the definition of the SPDE in section 4.2).

Assume that the spatial part was discretised as in Eq. (6.1.2) and expand the vector of random coefficients $\boldsymbol{u}(\boldsymbol{\omega})$ as

$$\boldsymbol{u}(\boldsymbol{\omega}) = \sum_{\beta\in\mathcal{I}} \boldsymbol{u}^{(\beta)} H_\beta(\boldsymbol{\omega}). \tag{6.2.1}$$

Each vector $\boldsymbol{u}^{(\beta)} = \big(u_1^{(\beta)},\ldots,u_n^{(\beta)}\big)^T$ belongs to one stochastic ansatz function and contains one coefficient for each spatial degree of freedom. The vector of all

unknowns is the block vector $\mathbf{u} = (\ldots, \boldsymbol{u}^{(\beta)}, \ldots)^T$. Together with Eq. (6.1.2), this is an expansion in tensor products of FEM ansatz functions and stochastic functions

$$u(x,\boldsymbol{\omega}) = \sum_{\beta\in\mathcal{I}} \boldsymbol{N}(x)^T \boldsymbol{u}^{(\beta)} H_\beta(\boldsymbol{\omega}) = \sum_{i=1}^{n} \sum_{\beta\in\mathcal{I}} u_i^{(\beta)} N_i(x) H_\beta(\boldsymbol{\omega}). \tag{6.2.2}$$

Once all coefficients $\boldsymbol{u}^{(\beta)}$ are computed, realisations of the solution may be generated cheaply (it is a a response surface; see section 6.3.1).

The techniques discussed in the following differ in how they compute the coefficients. But before presenting these, let us briefly discuss functions used for the ansatz functions $\{H_\beta\}$.

6.2.1 Ansatz Spaces

In principle, any set of linearly independent functions $\{H_\beta\}_{\beta\in\mathcal{I}}$ may be used in Eq. (6.2.1). But the stochastic discretisation is challenging due to the high number of dimensions. The numerical effort of many techniques increases exponentially with the number of dimensions, which makes them practically intractable for large dimensions. This exponential growth has been termed the "curse of dimensions" (e.g. Novak and Ritter, 1997; Novak, 1999), and discretisation techniques should be chosen so that this curse is avoided. Otherwise, they are only suited for the solution of problems in a small number of independent basic random variables $\omega_1, \ldots, \omega_m$.

Tensor Product Discretisations: The discretisations discussed here are tensor product discretisations. Recall that Ω_i denotes the range of the random variable ω_i and choose in each stochastic dimension Ω_i ($i = 1, \ldots, m$) linearly independent ansatz functions $h_1^{(i)}, \ldots, h_{p_i}^{(i)} : \Omega_i \to \mathbb{R}$.

Denote by $\beta = (\beta_1, \ldots, \beta_m)$ a multi-index (see section 2.2.3). The ansatz function indexed by β is then defined as the tensor product

$$H_\beta(\boldsymbol{\omega}) = h_{\beta_1}^{(1)}(\omega_1) \cdot \cdots \cdot h_{\beta_m}^{(m)}(\omega_m). \tag{6.2.3}$$

If $\mathcal{I}$ is the set of all appropriate multi-indices, then the approximation is

$$u(x,\boldsymbol{\omega}) = \sum_{\beta\in\mathcal{I}} H_\beta(\boldsymbol{\omega}) \boldsymbol{N}(x)^t \boldsymbol{u}^{(\beta)}. \tag{6.2.4}$$

Note again that the stochastic ansatz needs to be chosen such that each H_β has finite variance—or, more generally, is an element of a stochastic function space (S) discussed in section 4.1 that is appropriate for the SPDE at hand.

If the ansatz space contains all possible tensor products (this will be called *full tensor product ansatz*), then its vector space dimension $|\mathcal{I}| = \prod_i^m p_i$ increases exponentially in the number m of stochastic dimensions (if all $p_i > 1$). Full tensor product spaces are hence not suited for large stochastic dimensions.

Instead, it is common (e.g. Ghanem and Spanos, 1991b) to use for $p \in \mathbb{N}$ ansatz spaces of the form

$$\mathcal{I} = \{\beta \in \mathbb{N}_0^m \mid |\beta| \leq p\}. \tag{6.2.5}$$

Such sets are used e.g. in the polynomial chaos construction (see section 2.2.3), or in the Smolyak construction used in obtaining sparse grids (see section 7.1.4).

The vector space dimension of the ansatz space Eq. (6.2.5) is

$$\big|\{|\beta| \leq p\}\big| = \binom{m+p}{p}. \tag{6.2.6}$$

Due to the non exponential but only polynomial growth in the number of dimensions, this approach may be suited for high-dimensional problems.

(Generalised) Polynomial Chaos: This is a discretisation in multivariate polynomials. The ansatz functions are tensor products of univariate functions $h_k^{(i)}$, $k = 1, 2, \ldots, p_i$, chosen as polynomials of degree k. A convenient basis of the ansatz space is obtained if the $h^{(i)}$ are chosen mutually orthogonal in $L_2(\Omega_i; P_{\omega_i})$. Usually, the ansatz space is chosen as in Eq. (6.2.5).

If the random variables $\boldsymbol{\omega}$ are Gaussian, then the resulting ansatz space of multivariate polynomials is called the polynomial chaos (Wiener, 1938); see section 2.2.3. If the $\boldsymbol{\omega}$ are non-Gaussian, then the space of multivariate polynomials is sometimes called the *generalised polynomial chaos* (Xiu and Karniadakis, 2002c).

For example, if the $\omega_1, \ldots, \omega_m$ are mutually independent uniformly distributed, then an orthogonal basis is obtained by choosing $h_k^{(i)}(\omega_i) = l_k(\omega_i)$, where l_k is the Legendre polynomial of degree k. The more common case in the literature is that the $\omega_1, \ldots, \omega_m$ are independent centred Gaussian random variables with unit variance. Then an orthogonal basis is obtained by choosing $h_k^{(i)}(\omega_i) = h_k(\omega_i)$, where h_k is the Hermite polynomial of degree k.

For more general $\omega_1, \ldots, \omega_m$, the orthogonal polynomials may be obtained by numerical orthogonalisation. They may even be chosen such that they are at the same time orthogonal with respect to two scalar products (Babuška et al., 2002b), which simplifies the solution of certain problems; see section 6.4.8.

There is a large number of publications using the polynomial chaos or the generalised polynomial chaos; see section 6.4 and section 6.4.10.

Piecewise Polynomials: As for the finite element method, the ansatz functions may be chosen as tensor products of one-dimensional piecewise polynomials (Deb et al., 2001; Babuška et al., 2002b). Their supports form a regular mesh on $\Omega^{(m)}$ and approximations may be enhanced by either refining the mesh (h-method) or by refining the polynomial degree of elements (p-method). Approximation properties of such ansatz spaces are investigated by Babuška et al. (2002b); see section 6.4.9 where this approach is discussed in detail.

For regular meshes, the number of ansatz functions grows exponentially with dimensions. Hence, this approach is not suited for large dimensions of $\Omega^{(m)}$. This approach has been tested by Deb et al. (2001) for $m = 3$. Elman et al. (2002) used an ansatz of piecewise constant functions in higher dimensions ($m = 10$) for a deterministic operators with stochastic right hand side.

One way to realise piecewise polynomials in high dimensions might be sparse ansatz-spaces constructed by the same principle as the Smolyak quadrature formulas discussed above. Applications of sparse ansatz-spaces to (non-stochastic) PDEs can be found e.g. in Schwab and Todor (2002); Griebel et al. (1999).

6.3 Discretisations in Series Expansions

Once a series ansatz as in section 6.2 has been chosen for the solution, a set of equations for the unknowns needs to be obtained. Several techniques for this were proposed in the literature. Usually, not all of them are explicitly interpreted as variants of series expansions, but they are seen here in this unified manner.

6.3.1 Response Surface Techniques

Response surface techniques are often used to speed up Monte Carlo methods; see section 7.1 for a discussion of Monte Carlo techniques. Monte Carlo methods compute statistics of the response by solving many statistically independent realisations of SPDEs and by computing statistics of the sample solutions. Convergence proofs and convergence estimates of Monte Carlo methods for elliptic SPDEs are given by Babuška et al. (2002b).

The efficiency of Monte Carlo methods is sometimes enhanced by *response surface methods*: A random variable $u(\boldsymbol{\omega}), \boldsymbol{\omega} \in \Omega^{(m)}$ is approximated by sampling from it in a Monte Carlo fashion. It is then represented as a series expansion Eq. (6.2.1), where the coefficients $u^{(\alpha)}$ are obtained from the samples e.g. by a least square fit.

There also response surface used that are not series expansions of the type presented above. For example, response surfaces of the form $\{y|u(y) = 0\}$, which

specify polyhedrals or deformed balls, may be used to approximated failure surfaces in reliability investigations (Bucher et al., 2000).

Another way of representing a response surfaces might be interpolations with sparse grids (e.g. Barthelmann et al., 1999), but the author of the present report is not aware of work in this direction.

6.3.2 Perturbation Methods

Apart from the Monte Carlo method, perturbation approaches are probably the most popular techniques for solving SPDEs; see e.g. Kleiber and Hien (1992) for an introduction. Once the SPDE is approximated in a finite number of random variables, the stochastic influence is regarded as a perturbation around the system mean and moments of the response are obtained by a Taylor expansion of the solution in the basic independent random variables. Due to the Taylor expansion, this ansatz uses implicitly an expansion in multivariate polynomials, i.e. in polynomial chaos.

The resulting expressions are difficult to handle/obtain for higher moments. Therefore, usually expansions of degree two or smaller are used, and only the first and second order statistics are computed.

It is sometimes stated in the literature that the perturbation method does not take into account the distribution of the random variables, but of course it does so when moments are computed. The method may be expanded to compute moments higher than the second, but then complicated expressions result that are hard to handle in the general case. Another disadvantage is that the Taylor expansion permits only small deviations from the mean. Perturbation methods are only applicable to small coefficients of variance. According to Sudret and Kiureghian (2000), the C.O.V. should not exceed 20%.

A disadvantage is that the perturbation methods requires derivatives of the system matrix and of the right hand side with respect to the random variables. It is hence difficult to apply to existing software, or it requires to use software for automated differentiation.

This technique is applied e.g. by Haldar and Mahadevan (2000), who use perturbation approaches for reliability assessments and put special emphasis on the modelling of connections in steel structures and present a stochastic FEM method for stationary and instationary linear and nonlinear reliability analyses. Osnes and Langtangen (1998) apply a first order perturbation method, using the weighted integral method, to stationary stochastic groundwater flow (driven by D'Arcy's law) in two and three spatial dimensions. They state as an advantage of the weighted integral method that the 2nd order statistic of the response may be computed directly without discretising the random inputs. But in their approach the resulting number of random variables is equal to the number of finite elements, and a deterministic

problem has to be solved for each random variable. A discretisation of the random field e.g. by the Karhunen–Loève method might probably have reduced the number of random variables and probably might have saved work. Waubke (1996, p. 28f.) analyses a linear oscillator and demonstrates that the perturbation method is not well-suited for some dynamical problems.

6.3.3 Neumann Series

Solutions of SPDEs may be obtained as a *Neumann series*. The solution is obtained here as an expansion in multivariate polynomials, i.e. in polynomial chaos, and the coefficients in polynomial chaos are computed as the terms of a Neumann series.

Such methods are used e.g. by Ghanem and Spanos (1991b) and by Papadrakakis and Papadopoulos (1996). A rigorous analysis is given by Babuška and Chatzipantelidis (2002) for the elliptic SPDE Eq. (4.1.1). There, the property Eq. (4.3.9) is exploited to prove the convergence of the Neumann series.

Denote the operator Eq. (4.1.2) with the mean of $\kappa(x,\omega)$ as material by

$$Au := T_{\mu_\kappa} u = -\nabla \cdot \big(\mu_\kappa(x) \nabla u(x,\omega)\big) \tag{6.3.1}$$

and denote the operator with the fluctuating part $\kappa(x,\omega) - \mu_\kappa$ as material by

$$Bu = T_{\kappa_m - \mathbf{E}(\kappa)} u = -\nabla \cdot \Big(\sum_{i=0}^{m} \kappa_i(x) \xi_i(\omega) \nabla u(x,\omega) \Big). \tag{6.3.2}$$

Due to Eq. (4.3.9) (see Eq. (4.3.8) for the definition of σ_0),

$$c := \sup_{x \in R} \frac{\sigma_0(x;m)}{\mathbf{E}\left(\kappa(x)\right)} < 1, \tag{6.3.3}$$

which may be used to show that the Neumann series

$$U_K = \sum_{k=0}^{K} (-1)^k V_k, \quad \text{where} \quad V_k := (A^{-1}B)^k A^{-1} f, \tag{6.3.4}$$

converges to the solution u_m of Eq. (6.1.1) with an error in the energy norm (Babuška and Chatzipantelidis, 2002, Theorem 5.2)

$$\|U_K - u_m\|_{\mathcal{E}} \leq \frac{d\|u_m\|_{\mathcal{E}}}{1-c} c^{K+1}, \quad \text{with a constant } d. \tag{6.3.5}$$

It is obvious from Eq. (6.3.4) that U_K is a multivariate polynomial of degree K. Babuška and Chatzipantelidis (2002) show further, that each $V_k, k = 0, \ldots, K$ in

the Neumann series is a multivariate polynomial of degree k in the independent random variables $\xi_1, \ldots, \xi_m$ and can be obtained by solving m^k deterministic PDEs (e.g. by finite element techniques), each using the same operator $A = T_{\mu_\kappa}$, but with different right hand sides. Hence, the construction of U_K requires $1 + m + \cdots m^K = (m^{K+1} - 1)/(m-1)$ solutions of the same deterministic PDE with different right hand sides.

To speed up the convergence, Babuška and Chatzipantelidis (2002) present an improved Neumann scheme using a technique that is similar to the *stratification* technique used in Monte Carlo methods (see section 7.1.1): The finite dimensional probability space $\Omega^{(m)}$ is partitioned into smaller spaces, so that the restriction of the SPDE to each partition has a smaller constant c in Eq. (6.3.3). These SPDEs may then be solved in parallel on each partition.

As the effort grows only polynomially with dimensions m, this technique may be suited for the solution of high-dimensional problems (in many independent random variables).

6.3.4 Non-Intrusive SFEM

As mentioned, the $\{H_\beta\}$ are often chosen as orthogonal in $L^2(\Omega, P_{\boldsymbol{\omega}})$. In this case, the orthogonality of the ansatz-functions may be employed to compute the coefficients directly by orthogonal projections:

$$\boldsymbol{u}^{(\alpha)} = \mathbf{E}\left(\boldsymbol{u}(\omega) H_\alpha\right) \|H_\alpha\|_{L^2}^{-2}. \tag{6.3.6}$$

Ghiocel and Ghanem (2002) use this projection to solve an instationary SPDE modeling seismic soil-structure interaction "non-intrusively". Keese and Matthies (2003d) obtain solutions to a nonlinear stationary SPDE by orthogonal projection.

The expectation $\mathbf{E}\left(\boldsymbol{u} H_\alpha\right)$ may be computed by the high-dimensional integration techniques discussed in section 7.1. Ghiocel and Ghanem (2002) evaluate Eq. (6.3.6) by Monte Carlo with stratified sampling. Keese and Matthies (2003d) use Smolyak quadrature.

6.4 Galerkin Methods with (Generalised) Polynomial chaos

Another discretisation technique for series expansions are Galerkin methods. As the present review focus on these, they are presented in their own section.

Stochastic Galerkin methods with polynomial chaos ansatz spaces were proposed by Ghanem and Spanos (1990, 1991a). Their findings are summarised in their book (Ghanem and Spanos, 1991b). A summary of newer developments is

given by Ghanem (1999a), where a general purpose version of the stochastic finite element method for linear problems is proposed.

Ghanem and Spanos called their approach the *spectral stochastic finite element method*, as they used the stochastic Galerkin scheme in conjunction with a Karhunen–Loève expansion of the stochastic quantities and a *p*-method. But in the general presentation below, such methods will be called *Galerkin SFEM*.

Generalised polynomial chaos was also used in Galerkin SFEM techniques; see section 6.4.10. As polynomial chaos and generalised polynomial chaos are the same objects (just with respect to different measures), they are treated together in this section. Most statements for polynomial chaos made here are applicable to generalised polynomial chaos. As the resulting ansatz is a polynomial ansatz, the term "multivariate polynomials" would probably be preferable to avoid confusions.

Many publications on polynomial chaos Galerkin methods solve problems that may have no solution. For example, this is often the case, if SPDEs with Gaussian distributed material law are used. As discussed in section 4.3 this may lead to unsolvable SPDEs. Nonetheless, if Galerkin solutions of such constructions are compared to Monte Carlo simulations, good agreement is usually found.

Publications that use generalised polynomial chaos expansions often construct the material directly by its Karhunen–Loève series, as shown in section 3.2.2. This may be done so that the material is uniformly bounded from above and below. The marginal probability distribution of the material is then not known analytically (see section 3.2.2), but a well-posed problem is obtained.

If the ansatz is chosen as in Eq. (6.2.5), then the size of the ansatz space grows polynomially in the dimensions m. This makes the polynomial chaos ansatz in principle suited for moderate- to high-dimensional problems. Still, the number of ansatz functions grows fast for high polynomial degrees and/or for high dimensions. A way to overcome this may be adaptive methods; see section 6.4.6.

In publications on Galerkin SFEM, high-dimensional problems have rarely been tackled. Most publications discussed in the following sections test their numerical methods in less than five stochastic dimensions. However, it is not the randomness that makes stochastic problems difficult but the high dimensions.

6.4.1 Resulting Systems of Equations

If the ansatz Eq. (6.2.1) is inserted into the weak form of the SPDE and Galerkin conditions are applied, the following equations result.

- In the linear case Eq. (6.1.4), a linear system of block equations results: for

all $\gamma \in \mathcal{I}$
(6.4.1)
$$\sum_{\beta\in\mathcal{I}} \int_{\Omega^{(m)}} \boldsymbol{K}(\boldsymbol{\omega}) H_\beta(\boldsymbol{\omega}) H_\gamma(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \boldsymbol{u}^{(\beta)} = \int_{\Omega^{(m)}} \boldsymbol{f}(\boldsymbol{\omega}) H_\gamma(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega}),$$

where $P_{\boldsymbol{\omega}}$ denotes the probability distribution of the vector of random variables $\boldsymbol{\omega}$. These equations may be written in block matrix form as

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \tag{6.4.2}$$

Writing the integrals in Eq. (6.4.1) as expectations, the block matrix $\mathbf{K}$ and the block vector $\mathbf{f}$ have as entries

$$\boldsymbol{K}_{\beta,\gamma} = \mathbf{E}\left(\boldsymbol{K} H_\beta H_\gamma\right), \quad \text{and} \quad \boldsymbol{f}_\gamma = \mathbf{E}\left(\boldsymbol{f} H_\gamma\right). \tag{6.4.3}$$

In the literature on stochastic FEM (e.g. Ghanem and Spanos, 1991b; Ghanem, 1999b), the integrals in the stochastic dimensions are usually not computed exactly but by an expansion of $\boldsymbol{K}(\boldsymbol{\omega})$ in the ansatz space; see section 6.4.3.

- In the nonlinear stationary case Eq. (6.1.5), a system of nonlinear block equations

$$\boldsymbol{r}(\mathbf{u}, \boldsymbol{\omega}) := \int_{\Omega^{(m)}} r\Big(\sum_{\beta\in\mathcal{I}} \boldsymbol{u}^{(\beta)} H_\beta(\boldsymbol{\omega})\Big) H_\gamma(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega}) = 0 \tag{6.4.4}$$

results for all $\gamma \in \mathcal{I}$ (Keese and Matthies, 2002, 2003b,d), which may be written as a set of nonlinear block equations

$$\mathbf{r}(\mathbf{u}) = 0, \tag{6.4.5}$$

with a nonlinear function $\mathbf{r} : \mathbb{R}^{n\times|\mathcal{I}|} \longrightarrow \mathbb{R}^{n\times|\mathcal{I}|}$.

Solutions may be obtained by standard methods, e.g. by Newton or quasi-Newton methods (Keese and Matthies, 2002). This requires the evaluation of the high dimensional integrals in Eq. (6.4.4). For this, Keese and Matthies (2003d) used Smolyak quadrature; see section 7.1.

- For the instationary case Eq. (6.1.7), a set of ODEs results. The ansatz for the coefficients in Eq. (6.1.6) may be chosen as

$$\boldsymbol{u}(t, \boldsymbol{\omega}) = \sum_{\beta\in\mathcal{I}} \boldsymbol{u}^{(\beta)}(t) H_\beta(\boldsymbol{\omega}), \tag{6.4.6}$$

or, together with the spatial ansatz,

$$u(t, x, \boldsymbol{\omega}) = \sum_{i=1}^{n} \sum_{\beta\in\mathcal{I}} u_i^{(\beta)}(t) N_i(x) H_\beta(\boldsymbol{\omega}), \tag{6.4.7}$$

and the time dependent block vector of all unknowns is then

$$\mathbf{u}(t) = \left(\ldots, \boldsymbol{u}^{(\beta)}(t), \ldots\right)^T \quad \in \mathbb{R}^{n\times|\mathcal{I}|}. \tag{6.4.8}$$

The application of Galerkin conditions in the stochastic dimension yields a coupled system of ODEs

$$\mathbf{r}\left(\dot{\mathbf{u}}(t), \mathbf{u}(t), t\right) = 0, \qquad t > 0, \tag{6.4.9}$$

which may be solved by standard techniques.

There are some publications that solve linear instationary SPDEs with stochastic operator by Galerkin schemes (e.g. Waubke, 1996; Ghanem, 1998b,c, 1999c; Ghanem and Sarkar, 2000) and some publications that solve deterministic operators with stochastic right hand side (e.g. Jardak et al., 2002; Xiu and Karniadakis, 2002a); see section 6.4.10 and section 6.4.10. An analysis of the solution of stochastic initial value problems (ODEs with stochastic coefficients) was performed by Babuška and Liu (2003).

6.4.2 A Priori Estimates

The discretisation by global polynomials is a *p*-version of the finite element method in the stochastic dimension. Babuška et al. (2002b) give a priori estimates for the *p*-version as well as for the *hp*-version discussed in section 6.4.9.

The a priori estimates are not restricted to polynomial chaos, but were obtained for general piecewise polynomials. As ansatz space, Babuška et al. (2002b) use a full tensor ansatz containing all multivariate polynomials H_β with $0 \le \beta_i \le p_i, i = 1, \ldots, m$. Let $\boldsymbol{p} = (p_1, \ldots, p_m)^t$ and denote by $u_{k_0,\boldsymbol{p}}(x, \boldsymbol{\omega})$ the Galerkin approximation obtained for a spatial discretisation with linear finite elements on R with maximum diameter k_0 and with the stochastic ansatz space described above. Then the following a priori estimate holds for the L_2 error of the mean of the solution (Babuška et al., 2002b, Theorem 6.3):

$$\left\| \mathbf{E}\left(u(y,\cdot)\right) - \mathbf{E}\left(u_{k_0,\boldsymbol{p}}(y,\cdot)\right) \right\|_{L^2(R)} \le c\left(k_0^2 + \sum_{i=1}^{m} (c_i)^{2p_i+2}\right), \tag{6.4.10}$$

where $c, c_1, \ldots, c_N > 0$ are constants.

As this a priori estimate shows, for a fixed m, the *p*-version converges superlinearly, but the estimate does not give a useful estimate for $m \to \infty$.

In contrast to most other methods discussed in this section, the number of ansatz functions grows here exponentially in the number of dimensions, which makes the approach impractical for high dimensions. There, usually all H_β with $|\beta| \le p$ are used, but then this a priori convergence does not hold.

Benth and Gjerde (1998) compute convergence rates for approximations of an elliptic SPDEs depending on the stochastic regularity of the solution. They show that if the exact solution u is an element of $H_0^1(R) \otimes (S)^{-\rho,-q} \cap H^2(R) \otimes (S)^{-\rho,-q}, (\rho \in (0,1], k \in \mathbb{R})$ (see section 2.3 for definitions) and if $u_{m,p}$ denotes the solution obtained by a Galerkin approximation in tensor products of linear triangular finite elements with maximum diameter h and Wiener polynomial chaos of degree k in m Gaussian random variables, then the error in the solution can be computed by an explicit upper bound on the duality pairing with a smoother random variable $v \in H_0^1(R) \otimes (S)^{\rho,q}$, where $r = p - q$ satisfies $r > r^*$ as in Eq. (2.3.5). The estimate is

(6.4.11)
$$\left|\left\langle u - u_{m,p}, v\right\rangle\right| \leq c_1 \cdot \left(c(m,k,r)\|u\|_{H_0^1(R)\otimes(S)^{-\rho,-q}} + c_2 h \|u\|_{H^2(R)\otimes(S)^{-\rho,-p}} \right) \|v\|,$$

where $\|v\| = \|v\|_{H^1(R)\otimes(S)^{\rho,p}}$, where $c(m,k,r)$ is given in Eq. (2.3.6), and where c_1, c_2 are independent of u, v.

Note that $c(m,k,r) \to 0$ for $m \to \infty$ and hence, in contrast to the estimate Eq. (6.4.10), it yields a a useful estimate for $m \to \infty$. But here, the requirements are stronger than for the estimate Eq. (6.4.10).

6.4.3 Evaluation of the Integrals

The integrals occurring in the Galerkin scheme have to be evaluated. In the linear case Eq. (6.4.1), integrals

(6.4.12)
$$\boldsymbol{K}_{\beta\gamma} = \int_{\Omega^{(m)}} \boldsymbol{K}(\boldsymbol{\omega}) H_\beta(\boldsymbol{\omega}) H_\gamma(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega})$$

need to be computed (we will not discuss the integrals in the right hand side), and in the nonlinear case expressions

(6.4.13)
$$\boldsymbol{r}_\gamma = \int_{\Omega^{(m)}} \boldsymbol{r}\Big(\sum_{\beta\in\mathcal{I}} \boldsymbol{u}^{(\beta)} H_\beta(\boldsymbol{\omega})\Big) H_\gamma(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega})$$

must be evaluated.

Both integrals may be evaluated by the methods for high dimensional integration discussed in section 7.1. It was observed (Keese and Matthies, 2003d) that Eq. (6.4.13) may not be suited for the evaluation by Monte Carlo methods as the variance for the integrand may be large due to properties of the orthogonal polynomials used. In this case, sparse quadrature may be an efficient alternative (Keese and Matthies, 2003d,b).

In literature on SFEM, the integral Eq. (6.4.12) is usually not computed exactly, but the operator is replaced by an approximate expansion in polynomial chaos, e.g. see Ghanem (1999b),

$$\boldsymbol{K}(\boldsymbol{\omega}) = \sum_{\alpha\in\mathcal{J}} \boldsymbol{K}^{(\alpha)} H_\gamma(\boldsymbol{\omega}), \tag{6.4.14}$$

where $\mathcal{J}$ is a suitable set of multi-indices which may be chosen independently of $\mathcal{I}$. Once the matrices $\boldsymbol{K}^{(\gamma)}$ have been computed, the integral may be approximated by

$$\boldsymbol{K}_{\beta\gamma} = \sum_{\alpha\in\mathcal{J}} \boldsymbol{K}^{(\alpha)} \mathbf{E}\left(H_\alpha H_\beta H_\gamma\right). \tag{6.4.15}$$

The terms $\mathbf{E}\left(H_\alpha H_\beta H_\gamma\right)$ may be computed beforehand, and as they have a peculiar non-zero structure, the block matrix becomes a sparse block matrix, which permits efficient storage of the block equations (Ghanem and Kruger, 1996; Pellissetti and Ghanem, 2000; Matthies and Keese, 2001a,b, 2003).

Note that the sum in Eq. (6.4.15) may be seen to be a finite series (Matthies and Keese, 2003)–the reasons are that $H_\beta H_\gamma$ is a polynomial of degree $\beta+\gamma$ and that H_α is orthogonal on the space of polynomials of degree less than α. Hence, the set $\mathcal{J}$ may be chosen so that Eq. (6.4.15) is not an approximation but gives the exact value of $\mathbf{E}\left(KH_\alpha H_\beta\right)$.

The computation of the $\boldsymbol{K}^{(\alpha)}$ is demonstrated here for the elliptic example Eq. (6.1.3). They can be computed by the mutual orthogonality of the $\{H_\alpha\}$ as

$$\boldsymbol{K}^{(\alpha)} = \mathbf{E}\left(\boldsymbol{K}H_\alpha\right) = \int_R \int_{\Omega^{(m)}} \kappa(x,\boldsymbol{\omega}) H_\alpha(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \nabla\boldsymbol{N}(x) \nabla\boldsymbol{N}(x)^t\, dx \tag{6.4.16}$$

$$= \int_R \kappa^{(\alpha)}(x) \nabla\boldsymbol{N}(x) \nabla\boldsymbol{N}(x)^t\, dx, \tag{6.4.17}$$

where $\kappa^{(\alpha)}(x)$ is the projection of the random field $\kappa(x)$ on the polynomial chaos. The coefficients $\kappa(x)^{(\alpha)}$ can be computed analytically by Eq. (2.2.12). For a log-normally distributed $\kappa(x)$, the coefficients have additionally been derived from the characteristic function (Ghanem, 1999b).

A more efficient representation of the block matrix (Matthies and Keese, 2001a; Keese and Matthies, 2003f; Matthies and Keese, 2003), which is better suited for a parallelisation of the SFEM solver (Keese and Matthies, 2003c,e) is obtained if the random field is not projected onto the polynomial chaos directly, but if instead the uncorrelated random variables in its KL-expansion are expanded in polynomial chaos. One may then write

$$\boldsymbol{K}_{\beta\gamma} = \sum_i \sum_{\alpha\in\mathcal{J}} \xi_i^{(\alpha)} \boldsymbol{K}_i \mathbf{E}\left(H_\alpha H_\beta H_\gamma\right), \tag{6.4.18}$$

where the $\xi_i^{(\alpha)}$ are the polynomial chaos coefficients of the uncorrelated random variables occuring in the KL-expansion and where the $\boldsymbol{K}_i$ are the usual stiffness matrices obtained if the KL-eigenmodes are used as material.

6.4.4 Efficient Solvers for the Block-Equations

Ghanem and Kruger (1996) use the diagonal dominance in the block matrix to precondition a conjugate gradient solver and other Krylov solvers. An incomplete factorisation of the mean matrix of the system is used. Further, there and in Ghanem (1999a), the hierarchical structure is exploited using the Schur complement for prolongation, but it is not extended to multilevel solvers. The hierarchical approach is found to show a good performance.

Pellissetti and Ghanem (2000) extend the above findings; the data management of the block equations is addressed and an efficient matrix vector multiplication is implemented, which uses that each of the $\mathbf{K}_i$ has the same nonzero pattern. For preconditioning, inexact block Jacobi is proposed, and it is mentioned that for large random fluctuations this preconditioner may fail.

Matthies and Keese (2001a,b, 2002) use a larger class of block-diagonal solvers and additionally implement multilevel methods in the stochastic dimension.

Solvers for nonlinear stochastic PDEs were presented by Keese and Matthies (2002, 2003d,b) based on modified Newton methods and quasi-Newton methods.

6.4.5 Parallel Solvers

The potential for parallelisation of the method was observed in some publications.

Ghanem and Kruger (1996) suggested to parallelise or vectorise the matrix vector multiplications or to use a coarser parallelisation for the block matrix multiplication.

Xiu and Karniadakis (2002a) proposed to put each Karhunen–Loève mode on its own processor for speeding up the nonlinear solution of Navier Stokes equations with random boundary conditions.

Keese and Matthies (2003f,c,e) implemented a parallel solver for stationary stochastic problems with non-Gaussian material parameters. On 20 processors, good speedup was obtained for discretisations with more than a million unknowns.

6.4.6 Adaptivity and Sensitivity for Polynomial Chaos Ansatz Spaces

Almost all publications on polynomial chaos Galerkin schemes use the full set of polynomials in a given number of random variables up to a certain total degree.

As seen in Eq. (6.2.6), this results in a polynomially growing number of ansatz functions in the number of dimensions. The ansatz space is large, and hence adaptive methods are important.

Adaptivity for the response of a Duffing oscillator under stochastic Gaussian excitation is implemented by Li and Ghanem (1998). The time dependent excitation is Karhunen–Loève expanded in a set of random variables $\omega_1, \ldots, \omega_m$, and the response is expanded in polynomial chaos, where all polynomials of degree one are used and where higher order degree polynomials are only included on a subset of the most important random variables. This subset is found iteratively: Initially, the random variables most important for the expansion of the excitation are used. Then the system is integrated in time, and the subset of important random variables is chosen anew based on the norms of the projections of the solution on the random variables. This iterative scheme is repeated until the subset of most important random variables does not change any longer.

The results in the paper show good performance, but it is a heuristic procedure. The question whether the iteration leads to an improvement in speed when compared to starting with a larger ansatz space is not investigated. For the experiments, the system was solved up to four times until a stable subset of the random variables was obtained, but the procedure did not always converge. It might be possible to enhance the adaptation process by choosing not all polynomials on a subset of the random variables but by choosing a subset of the polynomials in all random variables.

Ghanem and Pellissetti (2002) compute the sensitivities of the stochastic response with respect to the data of the stochastic fields. This is performed by taking derivatives of the system equations with respect to the coefficients in one term of the Karhunen–Loève expansion on a certain element of the FEM mesh. For an instationary linear flow problem, the sensitivities of the solution on these coefficients are computed and it is proposed to use this methodology for refining the data available for the material parameters.

Keese and Matthies (2003a) implement an adaptive scheme for the solution of elliptic SPDEs, which is based on a goal-driven approach: It is observed that the ultimate goal in solving an SPDE is the evaluation of a functional of the solution, e.g. its mean at some position. The solution of a system dual to the original one can be interpreted as sensitivity of the system with respect to this functional. This sensitivity is successfully used to adaptively refine the ansatz space for the solution of the stochastic system.

This approach can be extended to instationary or nonlinear problems, and the error indicator obtained is directly applicable for space-stochastic adaptivity.

6.4.7 The Relation to Monte Carlo Formulations

Ghanem (1998c, 1999c) interprets the Karhunen–Loève-eigenmodes as scales of fluctuations upon which the stochastic systems acts as a nonlinear filter coupling the uncertainties from various scales. They identify the decomposition of a field into scales of fluctuations as a large advantage of their SFEM method. The disadvantage of Monte Carlo simulations, when compared to polynomial chaos approaches, is characterised there by the inability of Monte Carlo techniques to make use of the relation between different scales.

However, one may also use KL-expansions in the context of Monte Carlo or other integration techniques. Each KL-eigenvalue may be seen as a measures of the importance of the dimension associated with the random variable ω_i. When integrating over the probability space, one might take that into account and sample accordingly, e.g. by a Smolyak quadrature formula that uses less sampling points in the less relevant stochastic dimensions. For Monte Carlo techniques the efficiency does not decrease with the number of dimensions, hence it is probably not necessary to take into account relative the importance of the dimensions.

The polynomial chaos expansion may be coupled with Monte Carlo simulations (Ghanem, 1998a, 1999a, 1998b). The solution is expanded as $u(x,\omega) = \sum_\alpha u_\alpha H_\alpha(\omega) + \sum_i \hat{u}_i \delta(\omega - \omega_i)$, where δ is the Dirac function and where ω_i are the random events from the Monte Carlo simulation. A Galerkin projection is performed in the stochastic space (i.e. the residual is weighted both with the $H_\alpha(\omega)$ and with the $\delta(\omega - \omega_i)$. This results in a large set of block equations, where the part belonging to the Monte Carlo simulation is block diagonal.

From the mathematical viewpoint there is an open question to this approach: δ is a distribution (a linear functional) defined as $\mathbf{E}(\delta(\omega - \omega_i) f(\omega)) := f(\omega_i)$. For the weighting, the terms $\mathbf{E}\left(\delta(\omega - \omega_i)\delta(\omega - \omega_j)\right)$ are needed, but this expression is undefined for the functional δ. Further, the Delta distribution is not in the admissible space of the admissible stochastic space (S) of the SPDE; see chapter 4. Hence, in the opinion of the author, this approach may not be seen as a Galerkin scheme involving Dirac ansatz functions but as a variance reduction method: Ghanem (1998a) first obtains the polynomial chaos solution and then uses Monte Carlo simulations to reduce the error (which has a smaller variance than the solution). Alternatively, this approach might be used to compute error bounds for the polynomial chaos solution by Monte Carlo simulations.

If an approximation of the probability density of the solution is known, importance sampling may be performed. This is shown by Ghanem (1999a), where the exact characteristic function for second order polynomials in Gaussian RVs is inverted to obtain the probability distribution for the second order approximation of the process. However, if the stochastic dimensions are small, as in this example, it may be more useful to use another integration method than Monte Carlo integra-

tion, which is of greatest advantage for large stochastic dimensions. Nonetheless, an approximation of the probability density obtained by a Galerkin SFEM might a good a choice if used for importance sampling of a Monte Carlo integration in higher stochastic dimensions.

6.4.8 Doubly Orthogonal Polynomials

If the random material is represented as a series expansion in independent random variables

$$\kappa(x,\boldsymbol{\omega}) = \overline{\kappa}(x) + \sum_{i=1}^{m} k_i(x)\omega_i, \tag{6.4.19}$$

like in Eq. (3.2.4), then the equations for solving Eq. (6.1.4) resulting from the stochastic Galerkin method may be put into block diagonal form by choosing a special stochastic ansatz of multivariate polynomials (generalised polynomial chaos).

This is shown by Babuška et al. (2002b), who choose a basis of the (generalised) polynomial chaos consisting of polynomials, which are mutually orthogonal with respect to two different scalar products: Recall that P_{ω_i} is the probability measure induced by ω_i. In each stochastic dimension Ω_i choose a set of univariate polynomials $h_k^{(i)}$ of degree $k,\ k = 1,\ldots,n_i$ such that

$$\int_{\Omega_i} h_k^{(i)} h_l^{(i)}\, dP_{\omega_i}(\omega_i) = \delta_{kl}, \quad k,l = 1,\ldots,n_i, \tag{6.4.20}$$

and so that at the same time with some constants $c_{k,i}$,

$$\int_{\Omega_i} \omega_i h_k^{(i)} h_l^{(i)}\, dP_{\omega_i}(\omega_i) = c_{k,i}\,\delta_{kl}, \quad k,l = 1,\ldots,n_i. \tag{6.4.21}$$

The construction of this set of polynomials is possible for all $i = 1,\ldots,m$ and may be performed by solving a generalised eigenvalue problem (Golub and Van Loan, 1996, pp.463ff.).

The stochastic ansatz space is then constructed by tensor products Eq. (6.2.3) of these polynomials, which yields a block diagonal matrix in Eq. (6.4.2). The effort in solving this kind of problem with doubly orthogonal polynomials is the same as if a Monte Carlo simulation is used, where the number of experiments equals the number of multivariate polynomial ansatz functions. Babuška et al. (2002b) state that the computational complexity of this approach can compete with the Monte Carlo method.

This decomposition is only possible for linear SPDEs and for this special case of a random material. For nonlinear SPDEs and for general material parameters,

the block matrix cannot be block-diagonalised by this technique. Note that if the material is chosen like this, then its marginal distributions are not known analytically; see the discussion in section 3.2.2.

6.4.9 Stochastic hp-Galerkin method

The function spaces discussed above consist of tensor products of global polynomials in each stochastic dimensions Ω_i. Alternatively, piecewise polynomials H_α in the stochastic dimension may be used (Deb et al., 2001; Babuška et al., 2002b; Elman et al., 2002).

For each $i = 1, \ldots, m$, choose a one dimensional mesh with maximum diameter $k_i > 0$ discretising the stochastic dimension Ω_i. Denote by $h_1^{(i)}, \ldots, h_{n_i}^{(i)}$ piecewise polynomials on the mesh (e.g. usual one dimensional FEM shape functions). The ansatz functions H_α on $\Omega^{(m)}$ are then constructed for a multi-index α as tensor-products Eq. (6.2.3) of the one-dimensional piecewise polynomials.

Assume that the ansatz space $\{H_\alpha\}$ contains all multivariate piecewise polynomials with degree up to p on the regular mesh on $\Omega^{(m)}$ constructed as the Cartesian product of the one dimensional meshes and collect the maximum element size of the one dimensional meshes in the vector $\boldsymbol{k} = (k_1, \ldots, k_m)$ to denote the size of the m-dimensional regular mesh.

Assume that the spatial discretisation is performed by linear finite elements in R on a mesh with mesh size $k_0 > 0$. For the elliptic problem Eq. (4.1.1), the Galerkin approximation $u_{k_0,\boldsymbol{k},p}$ obtained for this spatial discretisation and the stochastic space described above can be shown to converge to the exact solution u (Deb et al., 2001; Babuška et al., 2002b).

A priori estimates are given by Babuška et al. (2002b, Remark 5.1) under the assumption that the solution has a certain stochastic and spatial regularity: Assume that for an integer $s \geq 1$ the solution $u \in C^{p+1}(\Omega^{(m)}; H^{s+1}(R) \cap H_0^1(R))$. Then the following a priori estimate holds

$$\| \mathbf{E}\left(u(\omega,\cdot)\right) - \mathbf{E}\left(u_{k_0,\boldsymbol{k},p}(\boldsymbol{\omega},\cdot)\right) \|_{L^2(R)} \leq c\left(k_0^{s+1} + k^{(p+1)(s+1)/s}\right), \tag{6.4.22}$$

where $k = max\{k_1, \ldots, k_m\}$, and where $c > 0$ is some constant. As this priori estimates shows, both the h-version and the p-version converge.

The method is tested by Deb et al. (2001) for a linear test-equation with piecewise constant H_α (p=0) on a rectangular mesh of hyper-cubes in $\Omega^{(m)}$ for uniformly distributed mutually random variables ω_i. Good agreement of the computed with the theoretical convergence rate is observed, but only small stochastic dimensions $m = 2$ are tested. Another test is performed in the same publication for the elliptic SPDE on a domain $R \subset \mathbb{R}^2$ and for the stochastic dimension $m = 3$. The computed solution is found to agree well with Monte Carlo simulations.

Only the *h*-version in the stochastic dimension has been tested in the publications. Elman et al. (2002) use this approach with a 10-dimensional stochastic space (with more than a million stochastic degrees of freedom) for an SPDE with deterministic operator and stochastic right hand side, which does not require the solution of block-equations Eq. (6.4.2), but many times the solution of one deterministic system.

This approach is impractical for stochastic operators in high stochastic dimensions due to the curse of dimension: as a regular mesh is used, the number of ansatz functions grows exponentially with dimensions; see Eq. (6.2.4). Applications to high dimensions require different computational techniques, e.g. sparse ansatz spaces might be a good candidate (Griebel et al., 1999; Schiekofer and Zumbusch, 1998; Schwab and Todor, 2002); see section 7.1.4. Additionally, adaptive techniques in the stochastic dimension may be required.

6.4.10 Applications of Polynomial Chaos SFEM

We give an overview of some selected applications. In the following, m is the number of stochastic dimensions and p is the total degree of polynomial chaos used in the publication.

Linear Stationary Problems: Most applications published so far are linear stationary problems. Here is a small selection.

Applications to stochastic beams and stochastic plates have been presented by Ghanem and Spanos (1991b). Ghanem and Kruger (1996) apply iterative solvers to a two-layered medium with a random interface and to a random plane stress problem example ($p = 4, m = 6$). The same example is solved by Ghanem (1998a) by a combination with a Monte Carlo method.

Pellissetti and Ghanem (2000) solve a cantilever beam with random stiffness and a pollutant transport problem resulting in nonsymmetric equations ($p = 3, m = 4, c.o.v. = 20\%$). Ghanem and Red-Horse (1999) solve a beam with random modulus of elasticity by spectral stochastic finite elements.

Instationary Problems: The following list is a selection of works using polynomial chaos for the stochastic discretisation of instationary problems.

Waubke (1996) computes stochastic response spectra of dynamic linear elastic soil problems with Gaussian or lognormally distributed stochastic shear modulus. The spatial discretisation is performed by a dimension reduction with Fourier or Hankel transforms or by boundary elements.

An instationary linear system of two coupled SPDEs describing the transport of a pollutant in unsaturated soil with random permeability is solved by Ghanem

(1998b). Linear instationary flow in a porous medium driven by D'Arcy's law on a two-dimensional rectangular area with a random hydraulic conductivity is solved in Ghanem (1998c). In both cases, the large set of resulting ODEs is solved with standard procedures ($m = 4$). This work is expanded by Ghanem (1999c) to the case where both the conductivity and the heat capacity are random (in the numerical experiments, the capacity is a random variable, only).

An axial bar with random Young's modulus and time-dependent excitation is solved by Ghanem and Sarkar (2000) by a frequency-domain transformation and projection on the prominent eigenvectors. The reduced model is solved with polynomial chaos.

Jardak et al. (2002) solve the advection equation with stochastic transport $\dot{u} + v(x,t,\omega)u' = 0$ by a polynomial chaos ansatz $u(x,t,\omega) = \sum_\alpha u_\alpha(x,t)H_\alpha(\omega)$. The polynomial chaos solutions are compared to exact solutions for simple cases and are found to match well. Both Gaussian and lognormal random fields are used.

Xiu and Karniadakis (2002a) solve the instationary incompressible Navier-Stokes equations with a deterministic operator but with random boundary conditions by a polynomial chaos ansatz. The Galerkin projection in the spatial and stochastic dimension results in a large coupled set of nonlinear ODEs which is solved in time by a third-order semi-implicit scheme. The spatial discretisation is performed by spectral *hp*-elements. For the simulation of a channel-flow with a Gaussian random velocity field on the boundary, a large correlation length was chosen so that an ansatz space of 15 stochastic ansatz functions was sufficient ($m = 2, p = 4$). Xiu et al. (2001) solve a fluid-structure interaction with stochastic inputs.

Li and Ghanem (1998) solve an instationary Duffing oscillator (deterministic operator) with stochastic Gaussian excitation by converting it into a system of ODEs and by expanding the response in polynomial chaos and applying Galerkin-conditions in the stochastic dimension.

The application of polynomial chaos-Galerkin schemes to Ito SDEs with colored noise for applications in option pricing has been proposed by Look (1998), but has not been implemented there.

Nonlinear Problems: There are a number of publications on nonlinear instationary systems under stochastic loading (see above). The number of publications on systems with nonlinear stochastic operator is scarce. Elasto-plastic problems have been solved by Anders and Hori (1999).

Keese and Matthies (2002) present efficient solvers based on modified-Newton and Quasi-Newton methods for the stationary solution of nonlinear flows with stochastic material parameters, and the question how to compute the required integrals in the stochastic space has been addressed. The computation of the integrals

has been proposed and implemented in Keese and Matthies (2003d) by Smolyak quadrature.

Applications of Generalised Polynomial Chaos: Xiu and Karniadakis (2002a), Lucor and Karniadakis (2003) and Xiu et al. (2002) propose to use functions from generalised polynomial chaos for a stochastic Galerkin scheme.

Xiu et al. (2002) and Lucor and Karniadakis (2003) use polynomial chaos to solve instationary coupled Navier-Stokes/structure equations (an elastically mounted cylinder in a laminar flow) with stochastic inputs (stochastic boundary conditions and stochastic forcings) and a deterministic operator. All random variables mentioned in the paper are Gaussian, and hence it is not clear whether more general stochastic ansatz functions than the original polynomial chaos have been tested. Time is discretised by an implicit 2nd order Newmark scheme, resulting in a large system of equations in every time step. The stochastic dimension is $m = 2$, polynomial chaos degree $p = 3$, and the ansatz space for the spatial dimension is large.

An elliptic operator with stochastic inputs is solved by Xiu and Karniadakis (2002b) using generalised polynomial chaos.

Xiu and Karniadakis (2002b) apply generalised polynomial chaos expansions to the solution of Poisson equations with stochastic operator and stochastich right hand side in probability spaces with various probability measures. The errors in the approximation are computed for a one-dimensional problem with known solution in one random variable, and it is found that the errors decrease exponentially in the degree of the polynomial chaos.

6.5 Other Approaches

We give a short (and incomplete) selection of approaches different to the ones discussed before.

6.5.1 Deterministic Operator with Stochastic RHS

The present review focuses on stochastic operators. For completeness, some recent results on deterministic operators with stochastic right hand side are mentioned.

While stochastic right hand sides are often forcing terms in instationary problems, for example wind loads (Walsh, 1984; Krée and Soize, 1986), the techniques described here do not consider this case but stationary SPDEs of the form

$$-\nabla \cdot (\kappa(x)\nabla u(x,\omega)) = f(x,\omega) \qquad x \in R.\ u|_{\partial R} = 0. \tag{6.5.1}$$

Linear systems under stochastic loads may be solved by linear filtering theory (e.g. Papoulis, 1991; Grigoriu, 1995) and are in general easier to solve than equations with stochastic operator. In this example, the mean of the solution u is the solution of the PDE when the mean of f is used as right hand side. Existence of solutions is analysed by Schwab and Todor (2002), Deb et al. (2001), and by Babuška et al. (2002b).

Deb et al. (2001) and Schwab and Todor (2002) show that the covariance $\mathrm{cov}_u(x_1,x_2)$ is the solution of a fourth order elliptic PDE on $R \times R$: It is the function that satisfies (Deb et al., 2001)

$$\mathcal{B}_C\big(\mathrm{cov}_u, v\big) = \int_R \int_R \mathrm{cov}_f(x_1,x_2) v(x_1,x_2)\, dx_1\, dx_2, \tag{6.5.2}$$

for all appropriate v on $R \times R$, where cov_f is the covariance function of the right hand side and where $\mathcal{B}_C$ is a bilinear form defined for suitable $u(x_1,x_2), v(x_1,x_2) : R \times R \longrightarrow R$ as

(6.5.3)

$$\mathcal{B}_C\big(u,v\big) = \int_{R\times R} \kappa(x_1)\kappa(x_2)(\nabla_{x_1} u(x_1,x_2))^t (\nabla_{x_1}\nabla_{x_2} v(x_1,x_2))\nabla_{x_2} u(x_1,x_2)\; dx_1\; dx_2.$$

Schwab and Todor (2002) show that for a coercive operator the PDE for the covariance has also a coercive bilinear form and provide a regularity analysis of cov_u in terms of cov_f.

For the numerical solution of Eq. (6.5.2), Deb et al. (2001) propose usual finite elements, while Schwab and Todor (2002) use multilevel techniques with sparse finite elements (see section 7.1.4).

Other methods for the solution of deterministic operators with stochastic RHS have been based on expansions of the solution in tensor products of finite element functions and stochastic ansatz functions. These are briefly addressed in section 6.4.10.

Elman et al. (2002) solve an accoustic scattering problem and use piecewise constant functions on a high dimensional regular mesh for the stochastic discretisation and obtain a set of linear equations with multiple stochastic right hand sides which they solve by iterative block algorithms. Other works have been based on expansions in (generalised) polynomial chaos (Lucor and Karniadakis, 2003; Xiu et al., 2002; Xiu and Karniadakis, 2002c,b; Xiu et al., 2001).

6.5.2 Semi-Direct Solution of SPDEs

Some problems may be solved exactly in the stochastic dimensions, once the spatial discretisation has been performed. Elishakoff et al. (1995) compute solutions for Bernoulli beams with stochastic flexibility by rewriting the beam bending equation as two separate equations, each with a stochastic right hand side. Exact 2nd

order statistics of the solution may then be obtained directly. The same principle is applied by Elishakoff et al. (1999) to Bernoulli beams with stochastic flexibility subjected to stochastic loads.

Elishakoff et al. (1997) generalise a method by Fuchs (1992) to stochastic beams: the finite element ansatz is represented in eigenmodes of the element stiffness matrices. This allows to assemble the global stiffness matrix $\boldsymbol{K}$ from the diagonalised element stiffness matrices so that it has the form $\boldsymbol{K} = \boldsymbol{R}\boldsymbol{D}(\omega)\boldsymbol{R}^T$, where $\boldsymbol{R}$ is a (non square) matrix containing deterministic coefficients and where $\boldsymbol{D}(\omega)$ is a diagonal matrix containing the random stiffnesses of the elements. By incorporating constraints from the essential boundary conditions, the matrix $\boldsymbol{R}$ may be replaced by an invertible matrix, and exact moments of the solution may be computed. If an expansion as in 6.2.1 is known for the material properties, exact solutions for $\boldsymbol{u}$ can be computed. The method has been presented by Elishakoff et al. (1997) for the Bernoulli beam, but it may be applied to more general linear problems.

This technique requires that the matrix $\boldsymbol{R}$ becomes invertible after eliminating the essential boundary conditions. For problems in higher spatial dimensions, this is usually not possible.

6.6 Comparisons of SFEM-Methods

	Monte Carlo	*h*-SFEM	*p*-SFEM
Work	Z/k_0^d	$\frac{(1+p)^m}{k_0^d k^m}$	$\frac{(1+p)^m}{k_0^d}$
$L^2(R)$ error	$k_0^2 + \frac{1}{\sqrt{Z}}$	$k_0^2 + k^{2(p+1)}$	$k_0^2 + r^{2(p+1)}$
$L^2(R)$ optimal work	$\mathrm{TOL}^{-(2+d/2)}$	$\mathrm{TOL}^{\left(-\frac{m}{2(p+1)}-\frac{d}{2}\right)}$	$\left(\log_r(\mathrm{TOL})\right)^m \mathrm{TOL}^{-d/2}$

Figure 6.6.1: A priori performance for the Monte Carlo method, the stochastic *p*-version and the *h*-version, taken from (Babuška et al., 2002b, Table 2). See below for the meaning of the symbols.

A priori convergence rates of the Monte Carlo methods, the *p*-version (polynomial chaos, see section 6.4.2) and the *h*-version of section 6.4.9 are compared by Babuška et al. (2002b). The findings are summarised in Table 6.6.1. There, the row labeled "Work" denotes the total amount of work for the given parameters. $L^2(R)$ error denotes the order of the a priori error for the $L^2(R)$ error of the solution's mean, while $L^2(R)$ optimal work shows the optimal work required to obtain a given tolerance TOL for the $L_2(R)$ error of the solution's mean.

In the comparison, it is assumed that the spatial problem has d dimensions and that its discretisation is performed by linear finite elements with mesh size k_0, which is the reason for the term k_0^2 in the $L^2(R)$ errors. The number of Monte Carlo simulations is Z. The polynomial degree of the stochastic ansatz functions is p, both in the h-version and in the p-version, and k denotes the maximum diameter in any stochastic dimension of the finite element mesh in the h-version, while m denotes the stochastic dimension, and r is some constant.

The table only shows the order of the effort involved. Which of the methods is favorable hence may depend on further constants in the estimates. As the optimal work of both the h-version and the p-version grows with the number of stochastic dimensions while the optimal work for Monte Carlo methods is dimension-independent, Monte Carlo methods are advantageous if the dimension is sufficiently high.

Xiu and Karniadakis (2002a) find the effort of solving Navier-Stokes equations with random boundary conditions by a polynomial chaos Galerkin-ansatz substantially faster than a standard Monte Carlo simulation (no explicit timings were given). It is claimed there that the solution of a stochastic flow around a cylinder, for which they used 6 days, would take more than a year via Monte Carlo simulation. Jardak et al. (2002) show that the solution of the advection equations with stochastic transport is sped up compared to a standard Monte Carlo simulation by some orders of magnitude (speed-up factors from 1000 to 200,000 are observed) if a polynomial chaos approach is used.

While these findings seem to demonstrate the superiority of generalised polynomial chaos approaches over Monte Carlo simulations, both Xiu and Karniadakis (2002a) and Jardak et al. (2002) use a small stochastic dimension ($m = 2$). The problem is hence unfavorable for Monte Carlo techniques. The comparison had probably been less favorable for the Galerkin methods if an integration technique more appropriate for small dimensions, like a full tensor product Gauss-quadrature rule, had been chosen in the comparison.

A comparison of some FEM-approaches was performed by Brzakala and Elishakoff (2001) for a test-equation with only one random variable,

$$K(\omega)u(\omega) = 1. \tag{6.6.1}$$

The random variable $K(\omega)$ was chosen there as $1+\alpha\omega$ with $\alpha \in (0,1)$ and ω a uniformly distributed random variable on $[0,1]$. Comparisons to the exact solution were performed for the perturbation method, a Galerkin-projection on Legendre polynomials in ω (generalised polynomial chaos), and a response surface method (1st-to 3rd-order polynomials). All tested methods worked well for small coefficients of variation (c.o.v. $\leq 43\%$). The generalised polynomial chaos expansion gave the best solutions in every experiment they performed. Based on their exper-

iments, Brzakala and Elishakoff (2001) concluded that all of these techniques are impractical for large coefficients of variation.

However, they used only polynomial expansions up to third degree, and the solutions converge to the correct solution if enough ansatz-functions are used. The errors in the results of all methods shown by Brzakala and Elishakoff (2001) behave qualitatively similarly with respect to the coefficient and variation and with respect to the degree of the expansion, which is not surprising as all these methods use a polynomial expansion.

6.7 Conclusions

This chapter discusses techniques for discretising the stochastic part of SPDEs. They all rely on representing an SPDE in a high dimensional space $R \times \Omega^{(m)}$. This high dimensional space exists implicitly in most publications on SPDE, but making the representation explicit was suggested by Deb et al. (2001); Babuška et al. (2002b); Babuška and Chatzipantelidis (2002).

The review has put the emphasis on stochastic Galerkin schemes as such schemes are the state-of-art for deterministic problems and as this has been an active area of research, recently. The a priori estimates in sections 6.4.2 and 6.4.9 as well as the comparisons in section 6.6 show that stochastic Galerkin schemes are a promising approach for the solution of SPDEs with stochastic operator.

Various discretisation techniques that are usually seen as independent of each other, namely some response surface methods, perturbation methods, Neumann series methods, and Galerkin methods, were viewed here as similar techniques: each represents the solution as a series of stochastic functions, but the coefficients of the series are computed differently. If the functions in the expansion are multivariate polynomials, then an expansion in polynomial chaos is obtained. Note that most publications use the term “polynomial chaos” to indicate an ansatz in orthogonal polynomials. But the polynomial chaos is formally defined as the span of certain polynomials, and the orthogonal polynomials are simply a convenient basis. Therefore, all global polynomial ansatz spaces were seen as polynomial chaos spaces in this review.

An advantage of the Galerkin methods over the other techniques is that it is easy to use higher polynomial degrees for the ansatz, that functions more general than polynomials may straightforwardly be used in the expansion, and that it is easier than in the other methods to refine the ansatz adaptively.

It was mentioned that the representation as a high-dimensional problem in principle permits to use the same ansatz spaces as in the finite element method. However, stochastic Galerkin schemes are challenging due to the high number of dimensions involved. Techniques applicable to high dimensional problems need

to be used and standard FEM techniques cannot be directly applied to high dimensional problems. A surprising finiding is that high-dimensional problems have hardly been tackled by Galerkin schemes in the literature on SFEM (often the stochastic dimension is smaller than five). The implementation of methods applicable to high dimensions remains a challenge.

From this point of view, comparisons between stochastic Galerkin schemes and Monte Carlo approaches found in the literature must be considered with care. They are only meaningful with respect to the number of stochastic dimensions used in the experiment. Statements about the superiority of Galerkin methods over Monte Carlo methods as discussed in section 6.6 may not be meaningful for higher stochastic dimensions (or for a different variance of the statistics considered), and a fair comparison would compare stochastic discretisation methods with integration techniques that are appropriate for the stochastic dimensions.

Another problem has been touched in section 6.6: the convergence of all methods discussed there degrades if the material parameters in the SPDE have a high coefficient of variation. For such problems, perturbation approaches show poor convergence as they rely on Taylor-expansions. Similarly, Monte Carlo methods converge slowly if the integrand has a high variance. High-dimensional quadrature techniques may be an alternative, but they were hardly used in SFEM techniques; see section 7.1.1. The polynomial chaos approach is in principle applicable even for large coefficients of variance, but in practice the required number of ansatz functions becomes too large. In publications, good results were obtained for c.o.v. of up to 50% (e.g. Ghanem, 1999b, 1998c). The treatment of problems, where the materials or the solution have a high coefficient of variance remains another challenge.

Mostly, publications on SFEM produce results in a theoretical setting. A validation of results is rare. Of the publications cited in this review, only Maglaras et al. (1997) performed a statistical validation and have good agreement of their results with experiments.

The experiments used in the literature to assess the quality of SFEM-techniques or to compare solution techniques must always be interpreted with respect to the stochastic dimensions and with respect to the c.o.v. used in the experiment. As well as one may construct experiments, where series expansions are superior to Monte Carlo and other integration techniques (small stochastic dimensions, high variance), one may also construct experiments where Monte Carlo techniques are favorable (high stochastic dimensions, low variance). It is thus not easy to state whether stochastic Galerkin methods are superior to Monte Carlo methods; the a priori convergence rates in section 6.6 may be a guideline when to use which method.

More open challenges in the context of stochastic finite elements include the solution of nonlinear problems. Only first steps in this direction have been taken

so far in stochastic Galerkin methods. Another problem that requires more work are adaptive techniques for choosing the stochastic ansatz.

Chapter 7

Numerical Procedures and Postprocessing for SFEM

This final chapter discusses special aspects of stochastic finite elements, like numerical procedures for high-dimensional integration, aspects of reusing existing software, and postprocessing of results.

7.1 Integrals in High Dimensions

The techniques of the previous sections require the evaluation of high dimensional integrals. For instance, random fields are discretised in section 6.4.3 by orthogonal projections as $\kappa^{(\alpha)} = \mathbf{E}(\kappa H_\alpha)$, and the Galerkin projections for nonlinear problems in section 6.2 involve integrals $\mathbf{E}\left(f(\kappa,u)H_\beta\right)$. In general, one needs to evaluate expectations (or integrals) of the type

$$\begin{aligned} I_m(\psi) &:= \mathbf{E}(\psi(\omega)) = \int_{\Omega^{(m)}} \psi(\boldsymbol{\omega})\, dP_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \\ &= \int_{\Omega_1} \cdots \int_{\Omega_m} \psi(\omega_1,\omega_2,\ldots,\omega_m) dP_{\omega_1}(\omega_1)\cdots dP_{\omega_m}(\omega_m). \end{aligned} \tag{7.1.1}$$

Several methods may be used for this. Their efficiencies depend on the number of dimensions m and on properties of the integrand. In reliability investigations high-dimensional integrals arise in the computation of failure probabilities, and often FORM or SORM methods (first/second order reliability methods) are used there, (e.g. Haldar and Mahadevan, 2000). This kind of integration shall not be treated here, where four classes of algorithms for high-dimensional integration will be discussed:

- Monte Carlo methods (e.g. Caflisch, 1998) are insensitive to the number of dimensions. But if the integrand has high variance or if a high accuracy is demanded, they require a high computational effort, see section 7.1.1.

- Quasi-Monte Carlo methods (e.g. Caflisch, 1998; Niederreiter, 1992) are discussed in section 7.1.2. They may be advantageous compared to Monte Carlo methods if the bounded variation norm of the integrand is small and if the number of dimensions is not too large.

- If a full tensor product of univariate quadrature formulas is used for the integration, the effort increases exponentially with the number of dimensions. Hence, such formulas are not suited for treating problems in a SFEM context; see section 7.1.3. The exponential increase of the effort in the number of dimensions has been termed the "curse of dimensions" (e.g. Niederreiter, 1992; Novak and Ritter, 1997; Novak, 1999).

- Smolyak type algorithms (Smolyak, 1963) combine tensor products of univariate quadrature formulas. They are well suited for high-dimensional problems if the integrand is smooth; see section 7.1.4.

Each of these methods obtains an approximation $Q_Z(f)$ of Eq. (7.1.1) by evaluating the integrand in Z integration points $\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(Z)} \in \Omega^{(m)}$ and by linearly combining the results with weights $w_1, \ldots, w_Z \in \mathbb{R}$,

$$Q_Z(\psi) = \sum_{i=1}^{Z} w_i \psi(\boldsymbol{\omega}^{(i)}). \tag{7.1.2}$$

7.1.1 Monte Carlo Methods

For an introduction to Monte Carlo methods see e.g. Sobol (1991), Fishman (1999), or the overview article by Caflisch (1998), and the references therein. A collection of recent research articles has been published by Schuëller and Spanos (2001).

Monte Carlo methods choose the integration points $\{\boldsymbol{\omega}^{(i)}\}$ as Z independent realisations of the random vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^t$ and use $w_i = 1/Z$. The integral is hence approximated as

$$Q_Z(\psi) = \frac{1}{Z} \sum_{i=1}^{Z} \psi(\boldsymbol{\omega}^{(i)}), \tag{7.1.3}$$

and the estimate $Q_Z(\psi)$ is a random variable converging a.s. to $I_m(\psi)$ due to Kolmogorov's strong law of large numbers. For large Z, the error $\varepsilon_Z := |\mathbf{E}(\psi) - Q_Z(\psi)|$ (a random variable) is

$$\varepsilon_Z \approx \sigma Z^{-1/2} \mathcal{N}(0,1), \tag{7.1.4}$$

where $\mathcal{N}(0,1)$ is a standard Gaussian RV and where σ is the standard deviation of ψ. The error is probabilistic and hence predictions can only be made with some confidence level. An alternative convergence estimate based on the *Koksma-Hlawka theorem* is discussed in the next section.

Due to the slow convergence of order $O(\sigma Z^{-1/2})$, evaluations with high accuracy require a high computational effort and a reduction of σ is important. Monte Carlo methods may be sped up by various techniques for variance reduction. A selection of common techniques (Caflisch, 1998) is briefly mentioned here: *Antithetic Variables* add integration points at $-\boldsymbol{\omega}^{(i)}$, which reduces the variance as the linear term of the Taylor expansion of $\psi(\boldsymbol{\omega}^{(i)}) + \psi(-\boldsymbol{\omega}^{(i)})$ around zero has zero expectation. *Control Variates* compute $Q_Z(\psi - \phi)$, where ϕ is a function with known $\mathbf{E}(\phi)$. *Matching Moment Methods* modify the sequence $\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(Z)}$ so that their statistical moments match the moments of the underlying distribution. *Stratification* computes the integral over $\Omega^{(m)}$ as the sum of integrals over disjoint sets partitioning $\Omega^{(m)}$ and may be enhanced by recursive application (Press and Farrar, 1990). *Importance Sampling* exploits that the integral may be written as $\mathbf{E}(\psi) = \int_{\Omega^{(m)}} [\psi(\boldsymbol{\omega})/p(\boldsymbol{\omega})] p(\boldsymbol{\omega}) dP_{\boldsymbol{\omega}}(\boldsymbol{\omega})$, where p is a probability density similar to ψ. This is then interpreted as integration of $\psi(\boldsymbol{\omega})/p(\boldsymbol{\omega})$ with respect to the probability density $p(\boldsymbol{\omega}) dP_{\boldsymbol{\omega}}(\boldsymbol{\omega})$, and the integration points are generated accordingly.

Monte Carlo simulations require reliable pseudo-random number generators, e.g. see Knuth (1981) for an introduction. Inadequate random number generators produce biased results (e.g. due to artificial correlations in tuples of pseudo-random numbers). As a finite state machine, every pseudo-random number generator repeats itself after some number of iterations. Upon this the error in the approximation ceases to decrease. Hence, a random number generator must produce independent tuples and have a large cycle length. On a parallel computer, the sequences in the individual processes also need to be mutually independent.

According to Caflisch (1998), the pseudo-random number generators presented by Press et al. (1997, chapter 7) are reliable. In a review of parallel random generators (Coddington, 1996) some packages are recommended for parallel random number generation, e.g. the SPRNG (Scalable Parallel Pseudo Random Number Generators) library (Mascagni et al., 1999), which permits long sequences and which is also recommended by Caflisch (1998).

7.1.2 Quasi-Monte Carlo Methods

Quasi-Monte Carlo methods evaluate the integrand in correlated points generated from so-called *low discrepancy series*. An important estimate for the upper bound in the error of an approximation of $I_m(\psi) = \int_{[0,1]^m} \psi(\omega) d\omega$ computed by

$Q_Z(\psi) = Z^{-1}\sum_{i=1}^{Z}\psi(\boldsymbol{\omega}^{(i)})$ from a series $\boldsymbol{\omega}^{(1)},\ldots,\boldsymbol{\omega}^{(Z)} \in [0,1]^m$ is the *Koksma-Hlawka theorem* (e.g. Caflisch, 1998, Theorem 5.1) which states that the integration error $\varepsilon = |I_m - Q_Z|$ is

$$\varepsilon \leq V(\psi) D_Z, \tag{7.1.5}$$

where $V(\psi)$ is the *total variation* of the integrand and where D_Z is the *discrepancy* of the series $\{\boldsymbol{\omega}^{(i)}\}$; see e.g. Caflisch (1998) for the exact definitions. Intuitively speaking, the discrepancy is the maximal error in approximating volumes of rectangular sets inside $[0,1]^m$ by using samples from the series. It is claimed by Caflisch (1998) that the total variation usually overestimates the error while the discrepancy of the series is usually a good indicator for the actual error. A sequence $\boldsymbol{\omega}^{(1)},\ldots,\boldsymbol{\omega}^{(Z)}$ is called *quasi-random* if its discrepancy obeys

$$D_Z \leq c(\log Z)^n Z^{-1}, \tag{7.1.6}$$

where c, n are constants which are independent of Z but usually depend on the dimension m. Often, $n = m$, and then the typical quasi-Monte Carlo error is obtained, which is $O(Z^{-1}\cdot(\log Z)^m)$. For high dimensions, the term $(\log Z)^m$ dominates, but nonetheless for many types of integrands a convergence rate of $O(Z^{-1})$ is obtained (Schürer, 2003).

A number of different quasi-random sequences have been developed, e.g. Halton's sequences (Press et al., 1997, Chapter 7.7), or Sobol sequences; see the monograph by Niederreiter (1992).

According to Caflisch (1998), quasi-Monte Carlo algorithms often converge faster compared to Monte Carlo in low dimensions, but for large dimensions their effectiveness reduces. For non-smooth integrands, they become less effective which may be cured by smoothing the integrand. Experiments by Caflisch (1998) for a Sobol sequence in four dimensions show an error of $O(Z^{-1})$, but in 16 dimensions, the error reduces to the Monte Carlo error of $O(Z^{-1/2})$. The applicability of quasi-Monte Carlo methods may be increased by dimension reduction techniques. A comparison to quadrature algorithms is given by Schürer (2003).

7.1.3 Quadrature by Full Tensor Products

Quadrature methods for high dimensions may be constructed as tensor products of one-dimensional quadrature formulas—e.g. of Gaussian, Clenshaw-Curtis, or Simpson quadrature formulas (e.g. Schwarz, 1993; Press et al., 1997). Assume that in each dimension Ω_i quadrature formulas $Q^{(i)}$ are given ($i = 1,\ldots,m$), each with the same number of nodes n and each exactly integrating polynomials of degree p with respect to the measure $dP_{\boldsymbol{\omega}_i}(\omega_i)$. The expectation $\mathbf{E}(\psi_i(\omega_i))$ of a

function $\psi_i : \Omega_i \to \mathbb{R}$ may be approximated as

$$Q^{(i)}(\psi_i) = \sum_{k=1}^{n} w_k^{(i)} \psi_i(\omega_k^{(i)}), \tag{7.1.7}$$

where $w_k^{(i)}$ are the weights and $\omega_k^{(i)}, k = 1, \ldots, n$ are the nodes of the quadrature formula $Q^{(i)}$.

A quadrature formula Q on $\Omega^{(m)}$ may be constructed as tensor product of the one-dimensional quadrature formulas, $Q = Q^{(1)} \otimes \cdots \otimes Q^{(m)}$. The integral $\psi : \Omega^{(m)} \to \mathbb{R}$ with respect to the measure $dP_{\boldsymbol{\omega}}(\boldsymbol{\omega})$ may then be approximated by

$$Q(\psi) = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_m=1}^{n} w_{k_1}^{(1)} \cdots w_{k_m}^{(m)} \psi\big(\omega_{k_1}^{(1)}, \ldots, \omega_{k_m}^{(m)}\big). \tag{7.1.8}$$

This tensor product quadrature formula is exact for all multivariate polynomials with (partial) degree not exceeding p, where a multinomial $\omega_1^{\alpha_1} \cdot \cdots \cdot \omega_m^{\alpha_m}$ is said to have (partial) degree p if $\alpha_i \leq p$ for all i.

The computation of Eq. (7.1.8) requires n^m evaluations of the integrand and hence is not feasible even for moderate dimensions—in $m = 30$ dimensions, more than a billion function evaluations would be required if $n > 1$.

For small stochastic dimensions m it was proposed (Ghanem, 1999a) to use Gauss-Hermite quadrature for the computation of the stochastic integrals in a polynomial chaos expansion in small dimensions and Monte Carlo integration in higher dimensions.

7.1.4 Smolyak Quadrature and Sparse Grids

Quadrature formulas based on Smolyak type combinations (Smolyak, 1963) of one-dimensional quadrature rules were applied successfully to high-dimensional integration, e.g. to 360-dimensional problems by Petras (2001, 2003). Other names for such constructions are *sparse grid methods*, *Biermann interpolation*, *Boolean methods*, *discrete blending methods*, or *hyperbolic cross points*, see the articles by Novak and Ritter (1996, 1999), and Gerstner and Griebel (1998) and the references therein. The following exposition is based on the latter two of these publications.

The Smolyak Construction: For the construction of a Smolyak type quadrature formula, not only one one-dimensional quadrature formula, but a sequence of quadrature formulas $Q_1^{(i)}, Q_2^{(i)}, Q_3^{(i)}, \ldots$ is required in each dimension Ω_i. For simplicity, assume that every method $Q_l^{(i)}$ of level l has polynomial exactness m_l

(independent of i) with $m_{l+1} \geq m_l$. Assume further that the $Q_l^{(i)}$ the same number of nodes n_l have for all i, where all lowest order methods use only one node, $n_1 = 1$. The set of nodes used by $Q_l^{(i)}$ will be called $\Xi_l^{(i)} = \{\omega_{l,1}^{(i)}, \ldots, \omega_{l,n_l}^{(i)}\}$ and the weights will be denoted by $w_{l,1}^{(i)}, \ldots, w_{l,n_l}^{(i)}$.

As discussed in section 7.1.3, tensor products of quadrature formulas are not feasible in high dimensions if they all use more than one node. But if the tensor product combines high order formulas in some dimensions with low order formulas in other dimensions, the resulting tensor product may still be practical in high dimensions.

For a vector $\boldsymbol{l} = (l, \ldots, l_m)^t \in \mathbb{N}^m$ construct a quadrature rule integrating functions $\psi : \Omega^{(m)} \to \mathbb{R}$ as

$$Q_{\boldsymbol{l}} := Q_{l_1}^{(1)} \otimes \cdots \otimes Q_{l_m}^{(m)}, \qquad \text{which is applied to } \psi \text{ by}$$

$$Q_{\boldsymbol{l}}(\psi) = \sum_{k_1=1}^{n_{l_1}} \cdots \sum_{l_m=1}^{n_{l_m}} w_{l_1,k_1}^{(1)} \cdots w_{l_m,k_m}^{(m)} \cdot \psi\big(\omega_{l_1,k_1}^{(1)}, \ldots, \omega_{l_m,k_m}^{(m)}\big).$$

As we required that the number of points in the first quadrature rule $n_1 = 1$, the evaluation of $Q_{\boldsymbol{l}}(\psi)$ is feasible even in high dimensions if only few $l_i \neq 1$.

The Smolyak construction combines such tensor product formulas. Let $Q_0^{(i)} := 0$ for all i and let

$$\Delta Q_l^{(i)} := Q_l^{(i)} - Q_{l-1}^{(i)}, \quad l \in \mathbb{N},\ i = 1, \ldots, m. \tag{7.1.9}$$

Then the level l Smolyak quadrature formula in m dimensions is

$$S_l^m = \sum_{\boldsymbol{l} \in \mathbb{N}^m,\, |\boldsymbol{l}| \leq m+l-1} \Delta Q_{l_1}^{(1)} \otimes \cdots \otimes \Delta Q_{l_m}^{(m)} \tag{7.1.10}$$

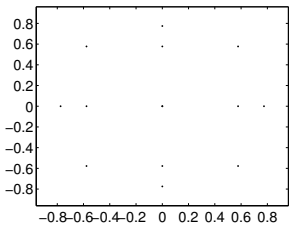
for integers $l \in \mathbb{N}_0$ (This is similar to the construction Eq. (6.2.5) for the polynomial chaos, but in Eq. (6.2.5) the smallest entry in a multi-index is zero while here all components in $\boldsymbol{l}$ are nonzero).

Eq. (7.1.10) can be rewritten (e.g. Novak and Ritter, 1996; Gerstner and Griebel, 1998) as

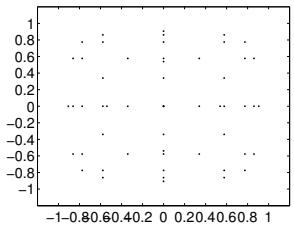
$$S_l^m = \sum_{\boldsymbol{l} \in \mathbb{N}^m,\, l \leq |\boldsymbol{l}|_1 \leq l+m-1} (-1)^{m+l-1-|\boldsymbol{l}|_1} \binom{m-1}{|\boldsymbol{l}|_1 - l} \cdot Q_{\boldsymbol{l}} \tag{7.1.11}$$

Note that sometimes another notation $A(q,m) := S_{q-m+1}^m$ is used (Novak and Ritter, 1996).

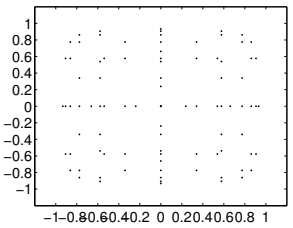
Sparse Grids: Every single tensor product formula $Q_{\boldsymbol{l}}$ evaluates the integrand on a regular mesh of nodes constructed from the nodes of the underlying univariate



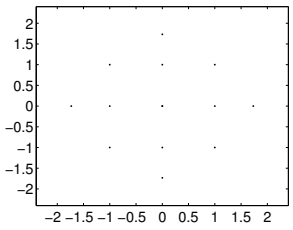
Gauss-Legendre formulas, S_3^2, 13 nodes.



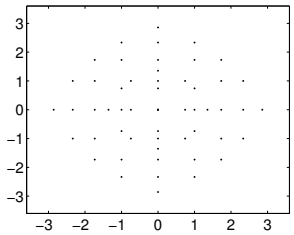
Gauss-Legendre formulas, S_5^2, 53 nodes.



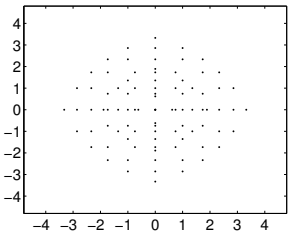
Gauss-Legendre formulas, S_6^2, 89 nodes.



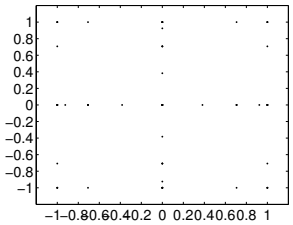
Gauss-Hermite formulas, S_3^2, 13 nodes.



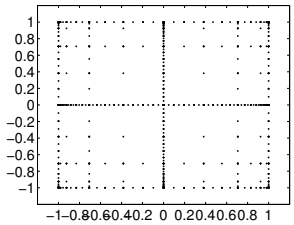
Gauss-Hermite formulas, S_5^2, 53 nodes.



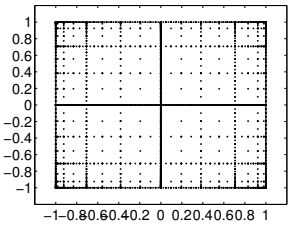
Gauss-Hermite formulas, S_6^2, 89 nodes.



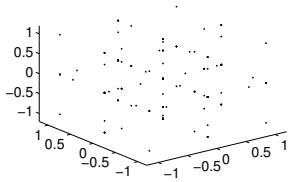
Clenshaw-Curtis formulas, S_4^2, 29 nodes.



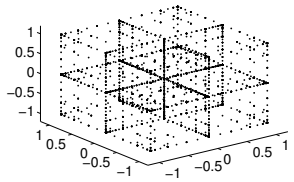
Clenshaw-Curtis formulas, S_7^2, 321 nodes.



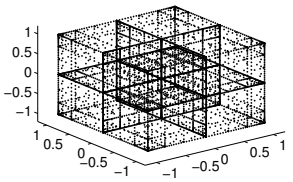
Clenshaw-Curtis formulas, S_9^2, 1517 nodes.



Clenshaw-Curtis formulas, S_4^3, 69 nodes.



Clenshaw-Curtis formulas, S_7^3, 1073 nodes.



Clenshaw-Curtis formulas, S_9^3, 5975 nodes.

Figure 7.1.1: Grids for Smolyak quadrature based on different one-dimensional formulas.

quadrature formulas $\Xi_{\boldsymbol{l}} = \Xi_{l_1}^{(1)} \times \cdots \times \Xi_{l_m}^{(m)}$. The Smolyak formula evaluates the integrand on the union Σ_l^m of these meshes,

$$\Sigma_l^m := \bigcup_{|\boldsymbol{l}|_1 \le m+l-1} \Xi_{\boldsymbol{l}} = \bigcup_{|\boldsymbol{l}|_1 \le m+l-1} \Xi_{l_1}^{(1)} \times \cdots \times \Xi_{l_m}^{(m)}. \tag{7.1.12}$$

If the one-dimensional quadrature formulas are nested, i.e. if $\Xi_{l+1}^{(i)} \subseteq \Xi_l^{(i)}$, then $\Xi_{\boldsymbol{l}} \subset \Xi_{\boldsymbol{l}'}$ whenever $l_i \le l_i', i = 1, \ldots, m$. This results in a smaller number of points compared to the non-nested formulas and hence in a reduced numerical effort. The resulting set of points Σ_l^m is called a *sparse grid*; see the examples discussed below and the plots for the Clenshaw-Curtis formulas in Fig. 7.1.1.

Smolyak formulas with nested grids may have better numerical stability than formulas on non-nested grids as positive and negative weights may partially cancel at common nodes (Novak and Ritter, 1996). Explicit formulas for the number of nodes in a sparse grid, and efficient algorithms for constructing it were presented by Petras (2003).

If the univariate quadrature formulas $Q_l^{(i)}, i = 1, \ldots, m$, are exact for all functions from spaces V_l, where $V_l \subset V_{l+1}$ for all l, then (Novak and Ritter, 1996, Theorem 2) the Smolyak quadrature formulas S_l^m are exact for all functions from the space

$$\sum_{\boldsymbol{l} \in \mathbb{N}^m,\ |\boldsymbol{l}| = l} V_{l_1} \otimes \cdots \otimes V_{l_m}. \tag{7.1.13}$$

- If the integrand is smooth, then Smolyak formulas based on one-dimensional Gauss formulas may be a good choice. The formula $Q_l^{(i)}$ is then chosen as the l-point Gauss formula corresponding to the measure dP_{ω_i} (Novak and Ritter, 1999).

  For example, if $\boldsymbol{\omega}$ consists of uniformly distributed random variables, Gauss-Legendre formulas may be used (see the first row of Fig. 7.1.1). If $\boldsymbol{\omega}$ is Gaussian, then Gauss-Hermite formulas may be a good choice (see the second row of Fig. 7.1.1).

  As l-point Gauss-formulas are exact for polynomials of degree of at most $2l-1$, Eq. (7.1.13) shows that the resulting Smolyak formula S_l^m is exact for all polynomials of total degree $2l-1$. Recall that a monomial $\omega_1^{\alpha_1} \cdots \omega_m^{\alpha_m}$ is said to have *total degree* p if $\sum_i \alpha_i = p$.

- Another common choice for the one-dimensional formulas are Clenshaw-Curtis formulas. The l-point Clenshaw-Curtis (CC) formula is based on an l-point Tshebyshev interpolation of the integrand and is exact for polynomials of degree not exceeding l.

The order of Clenshaw-Curtis formulas is lower than the order of Gauss type formulas, but CC formulas are often used (e.g. Novak and Ritter, 1996; Petras, 2003) as their nodes may be nested: If $Q_l^{(i)}$ is the CC formula in $n_i = 2^{i-1}+1$ points and if $Q_1^{(i)}$ is the 1-point CC formula, then the sequence of the one-dimensional quadrature formulas has nested points. Hence, a sparse grid results for the Smolyak formulas S_l^m (see the last row in Fig. 7.1.1) and S_l^m has total polynomial exactness $2^{l-1}+1$.

- Gauss formulas have the maximal polynomial degree of exactness with respect to the number of function evaluations, but as Fig. 7.1.1 demonstrates, their nodes are not nested. A variant are the Gauss-Patterson (or Patterson-Kronrod) formulas, which extend Gaussian quadrature formulas so that the resulting formula is maximal while at the same time the nodes are nested (Gerstner and Griebel, 1998).

  Nested quadrature rules for general probability measures are less developed than for the uniformly distributed case. Gerstner and Griebel (1998) state that for the Gaussian measure the Kronrod extensions are known only for a few special cases.

Even if the univariate quadrature formulas have positive weights, some weights of the Smolyak formula may be negative, but this usually does not lead to stability problems, as the absolute values of the weights stay relatively small (Novak and Ritter, 1997). The existence of negative weights requires techniques avoiding cancellations: the rounding errors may be reduced significantly by using a special summation order for the terms in the Smolyak formula (Gerstner and Griebel, 1998).

For nested grids, the weights associated with the nodes in the union of grids have to be computed, which may be expensive if done in a straightforward manner. The cost of computing these weights can be reduced drastically by observing that many weights are identical. By storing them in a tree structure, the weights for any node in the sparse grid may efficiently be retrieved (Petras, 2001).

Smolyak formulas that asymptotically use the minimal number of points were constructed by Petras (2003) by delayed (some univariate formulas are repeated) sequences of univariate formulas.

A comparison of adaptive and non adaptive interpolatory cubature rules with Monte Carlo and quasi-Monte Carlo methods is performed for dimensions up to $m = 100$ by Schürer (2003). It is discussed there that both the number of dimensions and the regularity of the integrand determines which integration method performs best. For a discontinuous test function, integration by adaptive quadrature formulas was found to be advantageous for dimensions up to 40. For a continuous

test function quasi-Monte Carlo methods were advantageous in all tested dimensions greater than five. For an oscillatory test function, adaptive quadrature based rules were found to beat quasi-Monte Carlo functions for all dimensions tested.

Finally note that Smolyak constructions may also be used for the interpolation of functions (Smolyak, 1963; Griebel and Knapek, 2000), for the construction of finite element ansatz spaces (Griebel et al., 1999; Schwab and Todor, 2002; Zumbusch, 2000), and for finite difference discretisations (Schiekofer and Zumbusch, 1998). Additionally, it is interesting to note that the stochastic finite element ansatz spaces discussed in section 6.2.1 are constructed similarly to the Smolyak construction.

7.1.5 Conclusions

Four different methods for integration in high dimensions were presented.

Of these, Monte Carlo methods are suitable if the variance of the integrand is small and if a low accuracy is required. Monte Carlo techniques do not take advantage of the smoothness of the integrand, and their advantage is their dimension independence. To be efficient they need to be combined with variance reduction techniques.

Quasi-Monte Carlo methods may often achieve a better convergence than Monte Carlo methods. They take low order smoothness into account. As Gerstner and Griebel (1998) state, quasi-Monte Carlo methods may be advantageous compared to Smolyak integration when the integrands are not smooth.

Smolyak constructions are well suited for smooth integrands, and Gerstner and Griebel (1998) state that they outperform both Monte Carlo and quasi-Monte Carlo for smooth functions, except for very high-dimensional problems.

7.2 Reusing Existing Software

When a probabilistic treatment of a problem is desired, often there already exists software treating the deterministic problem. It is highly desirable to reuse existing software. Software for stochastic analysis may be categorised into two categories (Bucher et al., 1999):

1. A Probability integrator may interface to an external FEM program, e.g.: ISPUD (by Bourgund and Bucher), or PROBAN (by Madsen et al.). These kind of software packages repeatedly pass realisations of system parameters to the FEM package and retrieve the results for a statistical analysis. For Monte Carlo simulations this is a natural approach, but it requires a lot of communication between both codes.

2. Integrated packages for stochastic structural analysis (SSA) do not interface to an external FEM program but provide their own, e.g. NESSUS (by Millwater) , CALREL (by A. Der Kiureghian), STRUREL/COMREL (Gollwitzer), SLANG (by Ch. Bucher).

Bucher et al. (1999) state that due to the large amount of communication involved, problems involving spatial structural randomness (random fields or processes) require an integrated SSA software, while systems with randomness in a small number of parameters can be implemented as a probability integrator interfacing to an external FEM package.

A coupling of the integrated program for stochastic structural analysis SLANG with ANSYS has been implemented by Bucher et al. (1999, 2000), where parallel execution of SLANG and ANSYS is done in a master slave fashion.

A framework for using existing finite element software in a black-box fashion for stochastic Galerkin schemes has been developed (for linear SPDEs) in Matthies and Keese (2001a, 2003), for nonlinear SPDE by Keese and Matthies (2003d,g) and in a parallel version in Keese and Matthies (2003f,c,e). This framework was coupled to ANSYS by Yu et al. (2003), see also the thesis by Yu (2003).

7.3 Visualisation of Polynomial Chaos Solutions

Not the coefficients of the solution but information postprocessed from it (i.e. functionals of the solution) are of interest to practicioners. This section discusses briefly how properties of the stochastic solution are computed and visualised in the literature. In principle, all common statistic, e.g. as described by Wackerly et al. (1996) or Lehmann (1999), may be of interest.

Visualisations found in publications on SFEM often show second order statistics of the solution, higher order statistics, cumulative distribution functions and/or probability density functions. In reliability analysis the probability of events is determined.

In the following section visualisation is only discussed for solutions given by a series expansion in the stochastic dimension (response surfaces). This covers most methods from chapter 6.

7.3.1 Second Order Statistics

Mean and covariance are simple to obtain for polynomial chaos expansion and are visualised quite often. For example:

For time dependent problems the evolution of isolines of the mean in a 2D domain are shown in Ghanem (1998c).

For vector fields, Xiu and Karniadakis (2002a) show the variance of the stochastic velocities along the channel's centerline for a 2D flow.

7.3.2 Functionals of the Solution

Ghanem (1998b,c, 1999c) visualise the stochastic solution of instationary systems by showing the coefficients of the solutions for various terms in the polynomial chaos, either over the whole spatial domain, or at various spatial locations over time. While this kind of visualisation shows the evolution of the polynomial chaos coefficients, it seems not to be well suited for the visualisation of statistical properties as it is hard to interpret the plots. This method can be interpreted as a visualisation of $\mathbf{E}\left(u(x_i,t,\omega)H_\alpha(\omega)\right)$.

7.3.3 Probability Density Function and Cumulative Density Function

The probability density function (PDF) of the approximate solution can be approximated by sampling from the solution (see section 7.3.4) in a Monte Carlo fashion. This may be performed cheaply once the response surface representation for the solution has been computed.

For an introduction to statistical probability density estimation see e.g. Martinez and Martinez (2002, chapter 8). The estimation by a histogram is usually a bad choice, but if a histogram is used, the width of bins should be chosen adequately, e.g. by Sturges' Rule. Methods more adequate than histograms are the following: *averaged shifted histograms*, which create many histograms with different origins and take their average; kernel density estimators, which smooth a histogram with a smoothing kernel; and *finite mixtures* which are a generalisation of kernel density estimators and approximate a PDF by a weighted sum of densities. Such techniques have been applied by Yu (2003) to visualise the displacements obtained for random elastic structures.

To obtain an analytical expression of the probability density of a random variable $\xi = g(\omega)$, the following three methods are common (Wackerly et al., 1996): The method of distribution functions computes $F_\xi(y) = P\{g(\omega) < y\}$ and than takes the derivative to obtain $f_\xi(y) = F'_\xi(y)$. If g has an inverse g^{-1}, the method of transformations computes $f_\xi(y) = f_\xi(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right|$. The method of moments uses the fact that two random variables ξ_1, ξ_2 have the same PDF if their moment generating functions $m_\xi(t) = \mathbf{E}\left(\exp(t\xi)\right)$ are equal for all t.

For polynomial chaos expansions of high polynomial degrees, none of these methods is feasible, as then the function g is not analytically invertible. For second

order polynomial chaos, the explicit characteristic function is given by Ghanem (1999a) and can be used to obtain the PDF for this case.

The PDF for a polynomial chaos approximation of a lognormal distribution is computed by Xiu and Karniadakis (2002a) by using the method of distribution functions. This method is only feasible if the roots of the polynomials can be found. Lucor and Karniadakis (2003) show the PDF of the pressure in a stochastic flow.

For expansions in polynomial chaos, Sudret and Kiureghian (2000) propose to compute the probability distribution by first order reliability methods: If $u = \sum u^{(\alpha)} H_\alpha(\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \Omega^{(m)}$, the value of the PDF $f_u(\hat{u})$ may be approximated by a first order reliability (FORM) analysis (see e.g. Haldar and Mahadevan, 2000), where the limit state function is chosen as

$$g(u(\omega)) = \hat{u} - u(\omega). \tag{7.3.1}$$

By computing the reliability index (finding the design point) and some standard computations of reliability theory, the PDF in $\hat{u}$ can be obtained. The expensive step is here the optimisation problem in an m-dimensional space. This technique has been applied by Yu (2003) to elastic stochastic problems, and methods based on sampling were found to be more robust and more efficient than this technique.

Other methods (Ghanem and Spanos, 1991b) for obtaining the probability density are based on Edgeworth expansions (also known as Gram-Charlier expansions).

A visualisation technique for visualising the cumulative density function of a stochastic field was presented by Yu et al. (2003) and in the master's thesis Yu (2003). It was observed that the CDF $F_{u(x)}(\hat{u}) = P\{u(x) \leq \hat{u}\}$ of a stochastic field $u(x,\omega)$ on a two-dimensional region $x \in R \subset \mathbb{R}^2$ is a function $F : R \times \mathbb{R} \to \mathbb{R}$ and may thus be visualised as volume data by standard visualisation techniques. A module for AVS (Advanced Visual Systems, 1989–2003) was implemented, which maps the CDF to a 3D-volume field, and isosurfaces of the CDF were shown.

7.3.4 Sampling from the Solution

A function expanded in polynomial chaos can be sampled from cheaply. Statistics may then be estimated from the samples by usual statistical methods. This is elementary textbook knowledge that may be found e.g. in the textbooks by (Papoulis, 1991) and by Wackerly et al. (1996).

7.3.5 Other Postprocessing

It is often important to evaluate the sensitivity of a solution with respect to the inputs. Ghanem (1999a) takes the derivative of the solution with respect to the the material properties using the Cameron-Martin shift (Malliavin, 1997).

The first passage statistics of a Duffing oscillator under random loading are computed by Li and Ghanem (1998) by a Monte Carlo simulation of the response process represented in polynomial chaos.

Often, the Karhunen–Loève eigenmodes and their eigenvalues are shown for inputs (e.g. Ghanem, 1998c, 1999c). The same technique might be used to visualise the solutions of stochastic systems, but the author is not aware of publications employing this technique for the solution.

Lucor and Karniadakis (2003) show the pressure distribution on a cylinder surface for a stochastic flow in polar plots.

Xiu et al. (2002) display the error bars of the pressure distribution on a cylinder in a flow under stochastic inputs, just as one would do for experimental results to show the tolerance of measurements.

A posteriori error estimates and sensitivities may be computed by dual techniques. This has been exploited by Keese and Matthies (2003a).

7.3.6 Conclusions

The visualisation of solutions to SPDEs is difficult as the solutions are high-dimensional functions. Usually second order statistics are visualised but little has been published on more special techniques.

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